BASIC STATISTICAL METHODS

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BASIC STATISTICAL METHODS

Education for Living Series

Under the Editorship of
H. H. Remmers

BASIC STATISTICAL METHODS

N. M. DOWNIE
Purdue University

R. W. HEATH
University of Arizona

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BASIC STATISTICAL METHODS

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EDITOR'S INTRODUCTION

> Until you have measured it, you don't know what you are talking about.
> *Lord Kelvin*

The clock, the calendar, high hopes, and deep convictions have for long been and still are to too great an extent the major criteria of educational achievement. Until the late J. McKeen Cattell in the late nineteenth century convinced Wilhelm Wundt that individual differences were worth systematic study in their own right, psychological study was aimed at establishing general laws of the "mind," and individual differences tended to be regarded as annoying anomalies, somewhat analogous to early disciples of Linnaeus ascribing species of living organisms that did not fit his binomial nomenclature to the work of the devil.

Mathematical invention and discovery in statistics applied to the biological and social sciences has seen phenomenal development, particularly in the last four decades. Statistics thus applied, however, is still largely not a part of the intellectual furniture of those charged with responsibility for educating the young—teachers and administrators—who, for varying reasons, tend to shy away from courses that would provide them with a functional modicum of this powerful tool for analysis and synthesis of data.

One of the reasons for this deficiency is doubtless a lack of mathematical knowledge and skills and hence a fear of the unknown in the form of difficult subject matter. Therefore a book that takes into account this lack of training and that approaches prospective teachers and administrators as they are is a most desirable addition to the literature of education. Without making unwarranted assumptions concerning the mathematical sophistication of the student, this text is in my judgment notably successful in presenting in simple, clear language the logic and content of statistics as needed by the beginning student of education and psychology. The introduction to distribution-free or nonparametric statistics—in my view destined to be much more used than heretofore—is particularly salutary, since the assumptions underlying parametric statistics are often dubious in the social sciences.

The introductory discussion of analysis of variance will lead the student to realize that for drawing inferences from data other powerful aids are available to those who wish to go on to more complete mastery of the tools requisite to more effective and valid control of the educational process.

Not the least valuable aspect of this book is its thorough consideration of the technology of tests and measurements, an area with which all students of education and psychology need to be concerned.

This book by my colleagues Professors Downie and Heath is a highly successful attempt to provide an effective learning aid. It affords me pride and pleasure to see it included in the series for which I have editorial responsibility.

H. H. REMMERS

PREFACE

Another statistics textbook? Those involved in the training of education, psychology, and sociology students are probably well aware that there are a number of sound and scholarly textbooks available in statistics. Why then should we prepare another one?

Despite the plenitude of texts available, we believe there exists a need for an introductory book which is as short as possible, as clear as possible, involves as little mathematics as possible, and stresses computation, application, and interpretation. We have tried to write such a book.

Because of the limited background and interest in mathematics of the typical student in the social sciences, no attempt has been made to derive the formulas or to impress the student with the beautiful mathematics underlying statistical theory. We have tried to present only enough theory so that the proper statistics can be chosen, computed correctly, and interpreted soundly. We have tried to do this briefly and clearly. To the extent that we have succeeded, this book differs from many existing texts.

The book is made up of three parts. The first nine chapters deal with descriptive statistics. The next six cover inferential statistics and the commonly encountered tests of significance. The last three chapters cover three topics not usually presented in the first course in statistics. These are (a) various correlation techniques, (b) test construction and theory (reliability, validity, and item analysis techniques), and (c) the more commonly used nonparametric techniques. Since the assumption of a normal distribution is so infrequently met in the social sciences, we feel that every student who studies elementary statistics should be aware of the possible use of these distribution-free statistics.

There are problems at the end of each chapter which are designed to offer practice in the techniques discussed in the chapter without requiring laborious computation. The answers to these exercises are given in the back of the book.

In writing this text we have incurred debts of gratitude to many individuals. We first wish to note the encouragement and advice received from Dr. H. H. Remmers who read the manuscript and then offered many constructive criticisms. We are also indebted to Dr. J. Francis Rummel for his valuable critical reading of the manuscript. Our drawings were prepared by George B. Rauch, Jr., and many of the calculations were checked by Richard D. Franklin. The entire manuscript was typed by Miss Jennie Goll. We are sincerely grateful for their patience.

Many sincere thanks are due also to various authors and publishers who gave us permission to use the materials which appeared in the tables in the Appendix. Special credit is given to them in footnotes that appear at the bottom of each table.

We are especially indebted to Sir Ronald A. Fisher, Cambridge, to Dr. Frank Yates, Rothamsted, and to Messrs. Oliver and Boyd, Ltd., Edinburgh, for permission to reprint Tables III, IV, and VI from their book *Statistical Tables for Biological, Agricultural, and Medical Research.*

N. M. DOWNIE
R. W. HEATH

January, 1959

BASIC STATISTICAL METHODS

1

INTRODUCTION

This book is written primarily for students in education, psychology, and sociology as an introduction to statistics.

The student certainly has the right to ask why he should take time to study this subject. We feel that in addition to the usual reasons, stemming from the almost universal belief in the value of a general education, there are at least five specific reasons why such students would want to study statistics.

1. *Daily Use.* Statistics are of immediate and practical utility. They help us get our work done quicker and better. They aid teachers in the assignment of grades and in the construction of tests. They help psychologists to interpret the measurements and observations they make. The sociologist finds them of similar use in evaluating the data he must work with.

In all the behavioral sciences, statistics have become a familiar part of the working day. To live and work in these fields with effectiveness and comfort requires a knowledge of the basic statistical methods.

2. *Problem Solving.* Often research is conducted on a limited scale, not to test theory, but to uncover information vital to the solution of a practical problem. We may be asked such questions as: "How can this test be improved?" "Which members of the group have the lowest morale?" "Does this method do a better job than that one?" "Is this variable related to another?" Such questions call for the application of statistical methods to the available data.

3. *Theoretical Research.* As the behavioral sciences have grown in sophistication they have become more quantitative. The development of theories which have a unifying effect on the information we gather compels us to test their predictions against measurements and observations. Theories of physical science which did not lend themselves to operational testing have long since been abandoned. The means by which we test the theories of education, psychology, sociology, and economics are largely statistical.

4. *Comprehension and Use of Research.* The competent practitioner, regardless of his field, must keep abreast of technical advancements. He must be able to read with understanding reports of problem solving and

theoretical research. In the behavioral sciences this means he must know what certain statistical terms mean and when particular methods should be used. Unfortunately, not all research is good research, and the burden of evaluation ultimately falls upon the reader. Statistics are not really the liars that they are reported to be. They can be misused and misinterpreted, but to those who understand them, they tell the truth or they say nothing.

5. *Enjoyment*. To many students it may seem that collecting and analyzing data would be anything but fun. Perhaps it is because such activities are frequently approached with an overwhelming pomposity and a deadly serious intent.

However, it does seem to be in the nature of men to enjoy such activities if they are left to their own devices. There is an almost sensual thrill that results when one finds that in an apparently chaotic and jumbled reality, he can find orderly and lawful relationships. When you successfully complete a puzzle, or a game of solitaire, you probably experience a dilute form of such an experience. When the pieces of the puzzle have social significance, and when the solution brings an understandable order to the behavior of people, the thrill is magnified many times. If the puzzles were too easy, they would be no fun. It would be a tragedy if we let their often exaggerated difficulties frighten us away.

WHAT WE CAN DO WITH STATISTICS

Teachers and social scientists are busy collectors of data of many different types. A large part of these data consists of measurements made with educational and psychological measuring instruments. A knowledge of statistics allows one to interpret these data correctly and, in addition, to obtain all of the possible information and relationships that might be present.

Let's spend a little time and think of some of the information that can be obtained from such data. Suppose that a teacher, a psychologist, or a sociologist has just made some measurements. What are some of the things that can be done with these results in order to get the maximum use and value from them? Here is a list of various possibilities.

1. We can determine averages. These averages give us a picture of the typical performance of the groups.
2. We can determine the variability of the measurements. We can use the average as our point of reference and determine how our scores or observations spread out about this central point.
3. We can make graphs, tables, and figures that will more clearly portray the nature of the group or groups.
4. We can change the raw scores into more meaningful types of scores. The most important of these are centiles or percentiles and standard scores. If we are dealing with classroom performance, we can change these raw scores into letter grades. If we are dealing with vocational tests we can convert them to job classifications. In sociological studies, we might want to have them yield indices of group membership.
5. We can determine the relationship of our measurements to measure-

ments made with another type of measuring device. The statistics used for this are called correlation coefficients and are among the most useful ones that we have. Some examples of what we can do are to find the relationship between intelligence test scores and classroom tests; between abilities and interests; and among various measures of physical development. Such variables as age and measures of physical development can be related among themselves and with our measures of achievement and with other psychological traits and sociological characteristics in many ways.

6. We can determine the reliability of our measuring instruments. This means taking two measurements on the same individuals with the same or different devices and finding the correlation between the two sets of data. As we shall see, there are also other ways of computing reliability coefficients.

7. We can determine the validity of our measurements. We will have to limit our discussion to statistical validity. This reduces again to computing the correlation between scores made on our test with another group of measurements, called criterion measures, to ascertain the relationship between the two sets of measurements. For example, intelligence tests are often validated by correlating scores on these tests with the students' grade point averages. If the intelligence tests are valid, it follows that those who obtained the highest scores on them receive the highest academic grades.

8. We can use one set of measurements, or combine them with others, to predict future status or behavior. This is probably the major end of all correlational work, for in themselves correlation coefficients are of little value.

9. We may base our measurement on the behavior of a sample of individuals, and on the basis of the statistics we have computed, we can make inferences about the larger population from which this sample was drawn. The drawing of statistical inferences is one of the chief functions and ends of modern research. The information about a small group or sample is usually in itself of limited use. We wish to be able to generalize on the basis of our research and make statements about the larger group.

10. We can compare performance on one variable with that made on another and see if there are significant differences in the results. Making these so-called tests of significance of differences is most important when we are conducting research on the introduction of a new method. Suppose that we are trying out a new reading technique. We have two groups, one of which we handle in the usual way in the course of a semester or a year, and an experimental one, which we treat in some new fashion. That is, the experimental group is the one that we apply the new reading technique to. At the end of the semester or of the school year, we administer the same reading test to both groups. We now examine the results made by both groups on this test. By inspection we cannot really tell whether or not they are different unless the difference happens to be a very large one. At this point we apply our statistics and make one or more tests of significance to see whether or not this difference is a real one or one that can be accounted for by chance variation. Making inferences and tests of differences makes up that part of statistics which is called *sampling* statistics. All the other methods mentioned so far are included in what we call *descriptive* statistics,

that is, they describe samples. Modern experimental research in all areas of knowledge from agriculture to zoology is based upon the proper use of sampling statistics.

It is hoped that the student will become more judicious in the use of statistics as a result of studying this book. In some circles statistics and statisticians have a bad name. It has been said that a person can find statistics to prove anything. Statisticians and liars have been lumped together in one breath. A recent book (**7**)[1] contains a good summary of the ways in which statistics are misused by individuals who wish to prove certain things or have special axes to grind. As will be shown as we go through this book, there is a time and place for every statistic. Certain ones are to be used when certain conditions prevail. We hope that the student will become aware of the conditions which underlie the use of each statistic. In summary, statistical method is the most powerful tool that we as professional practitioners have for arriving at verifiable truth.

It is not unlikely that many students beginning this course may have found mathematics unpleasant or difficult. Since this book is concerned with numbers and formulas they may think that it is complicated and involved. Such students should set their minds at ease.

You do not have to be a genius, or even fond of mathematics, to learn the methods presented here. The writers feel that if you have a grasp of the elements of seventh and eighth grade arithmetic, no trouble will be encountered. About the only further mathematical complication is the need to take square roots and to subtract negative numbers. A few of you will need to brush up a bit on these. A couple of hours should take care of this.

A more important problem for most students is that of precision and accuracy. It has been the writers' experience that most of the errors come about, not from using the wrong principle, but merely from carelessness in the simple operations of addition and subtraction.

Statistics, you will find, is one of those subjects that cumulates. One topic leads to another; the second is built upon the first. The work has to be kept up to date. If you try to build your knowledge of statistics on an incomplete foundation, it will surely topple.

Problems will be found at the end of each chapter. These will help you to test the stability of your progress.

---

[1] Boldface numbers refer to items in the References, p. 235.

2

A REVIEW OF ELEMENTARY THINGS

In this chapter we shall be concerned with two things: a review of the fundamentals of arithmetic and elementary algebra and a discussion of the fundamental nature of measurement. Many of the errors which occur in statistics are not caused because of lack of knowledge of statistics but by mistakes in very simple arithmetic. Hence we shall start by making a very rapid review of the rules to which we must be alert if our statistical computations are to be correct.

REVIEW OF ARITHMETIC

Decimals

Addition and Subtraction. In adding or subtracting decimals, the chief thing to keep in mind is to line the numbers up with the decimal point of one number directly under that of the one above.

To add 3.094, 235.67, and 45.7, we line them up like this:

$$\begin{array}{r} 3.094 \\ 235.67 \\ 45.7 \\ \hline 284.464 \end{array}$$

Multiplication. In multiplying decimals point off as many decimal places as there are in both the multiplicand and the multiplier taken together, as shown below:

$$\begin{array}{r} 1.072 \\ \times .02 \\ \hline .02144 \end{array} \qquad \begin{array}{r} .00007 \\ \times .2 \\ \hline .000014 \end{array} \qquad \begin{array}{r} 1.2 \\ \times 1.2 \\ \hline 1.44 \end{array}$$

Division. When two decimals are divided, the number of decimal places in the quotient is equal to the number of decimal places in the dividend minus the number of places in the divisor.

$$\frac{.012}{.3} = .04 \qquad \frac{2.0648}{.2} = 10.324 \qquad \frac{.008}{8} = .001$$

Fractions

Addition and Subtraction. When two or more fractions are to be added or subtracted, they must first be reduced to fractions having the same or a common denominator, as below:

$$\frac{1}{2} + \frac{1}{3} = \frac{3}{6} + \frac{2}{6} = \frac{5}{6} \qquad 2\frac{3}{4} + \frac{1}{2} = \frac{11}{4} + \frac{2}{4} = \frac{13}{4} = 3\frac{1}{4}$$

$$\frac{3}{8} - \frac{3}{16} = \frac{6}{16} - \frac{3}{16} = \frac{3}{16} \qquad \frac{x}{y} + \frac{a}{b} = \frac{xb}{yb} + \frac{ya}{yb} = \frac{xb + ya}{yb}$$

Multiplication. To multiply fractions multiply all the numerators by each other and place this over the product of all of the denominators, as shown:

$$\frac{1}{2} \times \frac{2}{3} = \frac{2}{6} = \frac{1}{3} \qquad \left(\frac{3}{4}\right)\left(\frac{2}{3}\right)\left(\frac{4}{6}\right)\left(\frac{4}{5}\right) = \frac{96}{360} = \frac{4}{15}$$

When multiplying fractions considerable time is saved if terms common to both the numerator and denominator are canceled out. This is the same as dividing the numerator and denominator of a fraction by the same number. The value of the original fraction is not changed. Let us rework the last example

$$\frac{\overset{1}{\cancel{3}}}{\underset{1}{\cancel{4}}} \times \frac{\overset{1}{\cancel{2}}}{\underset{1}{\cancel{3}}} \times \frac{\overset{1}{\cancel{4}}}{\underset{3}{\cancel{6}}} \times \frac{4}{5} = \frac{4}{15}$$

First we cancel the first 3 in the numerator with the 3 in the denominator. Then the first 4 in the numerator is canceled with the 4 in the denominator. The 2 in the numerator is divided into the 6 in the denominator, leaving a 3 in this place. Then we have in the numerator 1 × 1 × 1 × 4 and in the denominator 1 × 1 × 3 × 5. The product of all terms in the numerator is equal to 4; those in the denominator equal 15. Four-fifteenths is again our answer.

Division. To divide one fraction by another, invert the fraction which is the divisor and proceed as in multiplication.

$$\frac{3}{4} \div \frac{2}{3} = \frac{3}{4} \times \frac{3}{2} = \frac{9}{8} = 1\frac{1}{8}$$

$$2\frac{3}{4} \div \frac{11}{7} = \frac{11}{4} \times \frac{7}{11} = \frac{7}{4} = 1\frac{3}{4}$$

$$\frac{x}{y} \div \frac{a}{b} = \frac{x}{y} \times \frac{b}{a} = \frac{xb}{ya}$$

Negative Numbers

Addition. To add numbers each of which is negative, add the numbers in the usual fashion and place a minus sign in front of the sum

$$(-6) + (-8) + (-12) = -26$$

When the numbers are mixed and there are only two, that is, one negative and the other positive, subtract the smaller from the larger and give the remainder the sign of the larger.

$$\begin{array}{r} -6 \\ 8 \\ \hline 2 \end{array} \qquad \begin{array}{r} -22 \\ 28 \\ \hline 6 \end{array} \qquad \begin{array}{r} 56 \\ -72 \\ \hline -16 \end{array} \qquad \begin{array}{r} 19 \\ -30 \\ \hline -11 \end{array}$$

When there are more than two numbers of different signs, add all of the positive ones and then all of the negative ones and combine the results as above.

$$(-4) + (-7) + (8) + (13) + (-12) + (-5) = 21 + (-28) = -7$$

Subtraction. To subtract a negative number change its sign and proceed as in addition.

$$\begin{array}{r} 12 \\ -(-8) \\ \hline 20 \end{array} \qquad \begin{array}{r} -22 \\ -(-8) \\ \hline -14 \end{array} \qquad \begin{array}{r} -4.48 \\ -(-8.24) \\ \hline 3.76 \end{array}$$

Multiplication. When two numbers have similar signs, either positive or negative, the product of the two numbers is positive. When the sign of one number is positive and that of the other number is negative, the product of the two numbers is negative.

$$\begin{array}{r} 6 \\ \times 2 \\ \hline 12 \end{array} \qquad \begin{array}{r} -6 \\ \times(-2) \\ \hline 12 \end{array} \qquad \begin{array}{r} 6 \\ \times(-2) \\ \hline -12 \end{array} \qquad \begin{array}{r} -6 \\ \times(2) \\ \hline -12 \end{array}$$

Division. As in multiplication, when a number which is positive or negative is divided by a number of the same sign, the quotient is always positive. When dividend and divisor are of unlike signs, the quotient is always negative.

$$\frac{6}{2} = 3 \qquad \frac{-6}{-2} = 3 \qquad \frac{6}{-2} = -3 \qquad \frac{-6}{2} = -3$$

Use of Zero. The chief point to remember when using zero is that when any number is multiplied by zero, the product is always 0.

$$2 \times 0 = 0$$

$$(.5)(3.55)(0)(4976) = 0$$

When a zero is added to, or subtracted from a number, the number remains the same.

Square Root. In statistical operations, the student will find that there are many occasions on which he will have to be able to extract a square root. Table I in the Appendix contains the square root of all numbers from 1 to 1000. Unfortunately many of the numbers which we use do not come out even and, hence, cannot be read directly from the tables. We shall first illustrate the principle of taking a square root by the usual process. Suppose that we want to take the square root of a simple number like 144. Looking at the table in the Appendix we see that the square root of this number is 12. Suppose that we did not have a table. This is what we would do

$$\begin{array}{r|l} \multicolumn{2}{c}{1 \quad 2.} \\ \multicolumn{2}{c}{\sqrt{1 \quad 44.}} \\ \multicolumn{2}{c}{1} \\ \hline 2\ ? & 44 \\ \times\ ? & 44 \\ \hline \end{array}$$

Starting at the decimal point, we move to the left, making pairs. The last pair consists of only one digit, 1. We take the first pair and ask ourselves what is the largest square contained in it. In this case the largest square in 1 is 1. We write this under the 1 and subtract. Then we place the square root of this square over the first pair above the square root sign. We bring down the next pair of numbers. Then we bring the first number above the radical sign down and in so doing multiply it by 2, and we write this 2 to the left of 44. Then we ask ourselves what number can we place where the ? is and multiply the entire number by this and stay as close to 44 as possible, but not exceeding 44. This value is 2. A 2 is then placed where the question marks are, the multiplication is made, and the answer comes to 44. Hence 2 becomes the second digit in the answer and is written above the second pair of digits in the number. This time we have no remainder and the operation is over.

Let us this time extract the square root of 876.457.

$$\begin{array}{r|l} \multicolumn{2}{r}{2\ 9.\ 6\ 0\ 5} \\ \multicolumn{2}{r}{\sqrt{876.457000}} \\ \multicolumn{2}{r}{4} \\ 49 & 476 \\ \times 9 & 441 \\ \hline 586 & 3545 \\ \times 6 & 3516 \\ \hline 59205 & 297000 \\ \times 5 & 296025 \\ \hline & 975 \end{array}$$

Again, starting at the decimal point we make pairs. Going to the left, our first pair is 76 and the next is 8. Going to the right, the first pair is 45 and the next one is 70. Note that a zero was added here to make a pair. Beginning as before, the largest square contained in 8 is 4. A 4 is written below the 8, subtracted from 8, and the next pair, 76, is brought down. The square root of 4 is placed above the 8 and becomes the first digit in the answer.

Next this 2 in the answer is brought down and, as it is, it is multiplied by 2 and a 4 is placed to the left of 476. Then we look for the number which when placed to the right of this 4 and the whole number multiplied by this number will come as close to 476 as possible, but remain less than 476. The number is 9. So a 9 is placed as the second digit of our answer, 441 is placed below 476 and subtracted, and the next pair, 45, is brought down.

Now we bring down the two digits which appear in our answer and as we do we multiply them by 2 which gives us 58. Again by trial and error we see that a 6 will serve as our multiplier in the space to the left of 3545, and 6 becomes the third digit in the answer. Note that the decimal point in the answer appears directly above the point in the original number. We bring down the next pair, 70, and place it to the right of our remainder which gives us 2970.

This time we take the three digits in our answer and multiply them by two as we bring them down, and now we have 592. An inspection of the values at this point will show that there is no number that can be used this time since no matter what we multiply 592 by it will be larger than 2970. So we place a 0 in the answer and bring down the next pair. This next pair consists of a pair of zeros which we have added. It might be noted that we can add pairs of zeros as long as we desire and continue the extraction of the square root to many decimal places unless it happens to come out even at some point. Again we repeat the process, starting by multiplying the four digits in our numerator by 2 and continuing as above. This time a 5 appears in our answer and we stop the operation at this point.

It must be emphasized that it is most important when extracting square roots that the first step always be to point off into pairs and that this always be done by starting at the decimal point. Here are some more illustrations of pairing off

45678.9	04 56 78 . 90
4567.89	45 67 . 89
4.56789	04 . 56 78 90
45.6789	45 . 67 89

We can also get very close approximations of the square roots of numbers by using the square root table in the Appendix (Table I). Suppose that we extract the square root of 45360.9 using these tables. We enter the table and go down the column labeled N^2 until we come as close to 45360 as possible. The closest number in the table is 45369, and reading in the column to the left of this we find the square root to be 213. Next let us look up the square root of 453609. This time we go down the N^2 column until we come to 452929. Our value is about half-way between this one and the next one, and the square root is read as 673.5. It should be noted that when we take the square root of 4536.09 we proceed in the same way, going down the table until we come to 452929. The only difference is that we point off differently. In this case our answer is 67.35. In a similar fashion, the square root of 4.53609 would be approximately 2.13.

Exponents. We shall have limited use of exponents in elementary statistics, but the student should know what an exponent is and what to do with one. For example, in 2^3, the 3 is the exponent and it means to multiply $2 \times 2 \times 2$ or to raise 2 to the third power.

$$\begin{aligned} 3^2 &= 3 \times 3 = 9 \\ 4^3 &= 4 \times 4 \times 4 = 64 \\ x^4 &= (x)(x)(x)(x) \end{aligned}$$

Removing Parentheses and Simplifying. Sometimes we are confronted with a rather complex term such as the following and we have to simplify it. The general rule is to start first by removing the parentheses located on the inside.

$$\begin{aligned} &[(12 + 4)4] - [(3 + 10) + (6 \times -12)] \\ &= [(16)4] - [(13) + (-72)] \\ &= 64 - (-59) \\ &= 64 + 59 \\ &= 123 \end{aligned}$$

Proportions and Percentages. A proportion, the symbol for which is p, is defined as a part of a whole. If a pie is cut into six equal parts and you are given one of these parts, you have received one-sixth of the pie. That is your proportion and we can write that $p = {}^1/_6$ or .167.

To use another example, suppose that in a given class of 40 students, 4 receive A's as their final grade; 10, B's; 15, C's; 7, D's; and 4, F's. What proportion of each letter grade was given?

			p	P
A	4	$\frac{4}{40} =$	.10	10
B	10	$\frac{10}{40} =$	.25	25
C	15	$\frac{15}{40} =$	.375	37.5
D	7	$\frac{7}{40} =$	.175	17.5
F	4	$\frac{4}{40} =$	.10	10
			1.000	100.0

It should be noted from the above that the sum of the proportions for a given example is always 1 and the maximum value that any proportion may have is 1.

Percentages are obtained by multiplying a proportion by 100. The symbol for a percentage is P. The corresponding percentages for the above table are shown in the column at the right. It will be noted this time that the sum of the percentages for our data add up to 100.

Rounding Numbers. In rounding numbers to the nearest whole number or to the nearest decimal place, we proceed as follows:

To the nearest whole number:

7.2 = 7
7.8 = 8

To the nearest tenth:

7.17 = 7.2
7.11 = 7.1
.09 = .1

To the nearest hundredth:

7.177 = 7.18
.674 = .67
1.098 = 1.10

In all of the above the general rule is that if the last digit is less than 5, it is dropped; if the last digit is more than 5, the digit preceding the last one is raised to the next higher digit. The only complication arises when numbers end in 5. There is a general rule for this case. When the digit preceding the 5 is an odd number, this digit is raised to the next higher one; when it is an even number, the 5 is dropped. The following illustrate this rule:

8.875 = 8.88
8.05 = 8.0
5.25 = 5.2
66.975 = 66.98

Significant Digits. The question frequently arises in rounding numbers as to how many digits we should have in our answers. The general rule is that one should have no more significant digits in his answer than there are significant digits in the original data. For example, if we had a series of test scores all of which contained two digits, then we should have two digits in the average or mean which we compute from these data. The usual practice, however, is to carry such figures to the nearest tenth. There is nothing to be gained in computing these averages to five or six decimal places. No meaningful accuracy is obtained from these large decimals. As a matter of fact such large decimals mean nothing when computed on the basis of two-place numbers. A good rule to follow is to have in answers one more significant digit than was present in the original numbers. Here are some examples of the number of significant digits in a series of numbers

78	Two	1008	Four
786	Three	1976.09	Six
78.2	Three	.0025	Two (the two zeros merely point off the number in this case)
106.09	Five		

Sometimes students get into trouble as a result of rounding numbers in their problems too freely. Suppose that we have an operation which consists of six distinct steps. At the conclusion of the computations for each step, the student rounds his results. A series of a half a dozen such roundings in the course of the solution of a problem causes inaccuracies to enter the work. If we are working a problem and are going to express our answer to the nearest tenth, a good rule to follow is to carry all operations through in terms of hundredths and to round to the nearest tenth in the last step.

Statistical Symbols. In any book like this one the reader is going to be confronted with a large number of symbols which are strange to him. The easiest thing to do is to learn these as they appear. Some of them are likely to be new, but many are merely the first letter of the various statistics. Unfortunately, there is no general agreement among statistical writers as to exactly what symbol stands for the various statistics. Those used in this book are the ones frequently encountered in the social sciences at the present time.

HOW NUMBERS ARE CONSIDERED IN STATISTICS

Types of Measurements. We can classify our data into two types: continuous and discontinuous, or discrete. Feet, pounds, minutes, and meters are examples of continuous data. With scales for any of these we can make measurements of varying degrees of fineness. For example, we can break meters into centimeters, these into millimeters, and with intricate devices we can make our measurements finer and finer. Such data can be considered as being on a straight line and the size and accuracy of the measurements that we can make along this line depend on the way that the measurements are made.

To illustrate, suppose that we measure a boy and we say that he is 57 inches tall. Does this mean that he is exactly 57 inches tall? Probably not. In reading our scale we merely took our reading to that number in inches to which the boy's height was closest. This 57 inches includes then a band of our line; that is, it extends from 56.5 to 57.4999 inches. We can round the latter and then say that 57 includes everything from 56.5 to 57.5 inches. Similarly 58 extends from 57.5 to 58.5. We state that each number which represents continuous data has a lower limit and an upper limit, as shown in Figure 2.1.

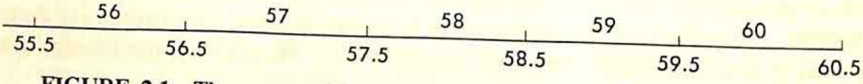

FIGURE 2.1 The upper and lower limits of units classified as continuous data.

Discontinuous or discrete data, on the other hand, are based upon measurements which can only be expressed in whole units. The number of people, for example, can only increase in whole units in contrast to measurement made in feet which can be broken down into smaller and smaller units. Other examples of discrete units are number of words spelled correctly, number of objects assembled, and the number of cars passing a point during a certain period of time. The student will note, however, that in statistical work all data tend to be treated as continuous and we end up by making such statements as that the typical graduate of College A has 2.8 children. In this book all data will be so treated, and the student should as soon as possible get accustomed to thinking of every number as having an upper and lower limit.

In measurement work it is customary to describe four types of measurements: *nominal*, *ordinal*, *interval*, and *ratio* **(15)**. As these are listed, the lower-level type occurs first. When a *nominal* scale is used, a number is used to designate a class or a category. For example, a high school population may be classified in four groups: Group 1, freshmen; Group 2, sophomores; Group 3, juniors; and Group 4, seniors. Another example of such a scale is setting up two groups on the basis of sex. After the groups are set up, the numbers in each group can be determined and the simplest types of statistics used.

When measurements are arranged on an *ordinal* scale, they are placed in rank order, starting with either the largest or the smallest one first. When data are on such a scale, simple statistics such as the median, centiles, and the Spearman rank-order correlation coefficient may be used.

The next highest type of scale is the *interval* scale. On scales of this type we find equal units of measurement. The commonly used Fahrenheit or centigrade thermometers are of this type. The final type of scale, the *ratio* scale, has in addition to equal units an absolute zero. Such a scale is the Kelvin scale. Its absolute zero is the point at which there is the absence of heat, or −273°C. Measurements made in feet, pounds, gallons, and the like are also on a ratio scale.

When data are in terms of feet we can say that one length is twice or half the size of another. When our measurements are on an interval scale we cannot do this and make sense. For example, suppose that the maximum temperature today is 60°; the same day last year it was 30°. In this case we cannot state that it is twice as warm today as it was on the same date last year. What is the difference between these two conditions? When we were dealing with feet, we were using a measuring scale that is based upon an absolute zero; in the second case we are using a scale which starts 32 degrees below the freezing point of water. When measurements are on a ratio scale meaningful comparisons can be made. As a matter of fact, when data are of this type, all of the usual mathematical and statistical manipulations may be made. However, in actual practice, many of our measurements are based upon an interval scale and we apply practically all of our statistical techniques to these measurements.

What can we say about the measurements that we make in education, sociology, and psychology? First of all, we assume that they are made up of equal units of measurement. An inspection of certain of these, such as intelligence quotients, leads us to wonder whether or not this happens to be true. And, also, our scales do not possess an absolute zero. The physicist can describe his absolute zero on his heat scale. It is not difficult to visualize zero inches, pounds, or meters. But what does 0 IQ mean? Or what does it mean when a boy gets a score of 0 on a geography test? Actually we do not know what these scores of 0 mean. Then it follows that we have no basis for claiming that a child with an IQ twice the size of that of another child is twice as bright or that the child whose score on an arithmetic test is twice that of another child knows twice as much arithmetic as the first child does.

SAMPLES AND POPULATIONS

It is important that we distinguish between these two terms. To do this let us use an illustration. Suppose that we are interested in the mental ability of children in the second grade. One way to investigate this is to give intelligence tests to second graders. We begin by obtaining permission to administer the test to one group of second graders. We compute an average or mean score for the group. This mean score is a statistic and gives us the average performance of our sample. Since there are so many second grade classes we could keep this process going for a long time by drawing sample after sample. If we combine all of these sample averages or means, and get a grand mean from this, this average will be the best estimate obtainable of the average intelligence of all the second graders. That is, the average of all of the values of our samples is used to tell us something about the population value. All second graders in the United States make up the population or universe from which the various samples are drawn. Values which refer to populations are referred to as *parameters* in contrast to *statistics* which are reserved to describe samples. Populations, as the term is used in statistics, are arbitrarily defined groups. They need not be as large as the one used here as an illustration. We could let the 552 seniors in a certain school system be our population and from this we could draw samples as described above. One of the major aspects of statistical research is making inferences about population characteristics on the basis of one or more samples that have been studied.

Exercises

1. Subtract the following:

(1)	26.09	(b)	87.54	(c)	.0987	(d)	14.67
	7.76		−22.13		−.9987		32.53
(e)	−.0987	(f)	−54.89	(g)	1.0987	(h)	67.87
	.9872		−6.76		11.6009		−7.13

2. Multiply the following:

(a)	45.67	(b)	12.345	(c)	.0008	(d)	−11.098
	8		.006		.07		.06
(e)	456.89	(f)	−.0768	(g)	−.777	(h)	−6.578
	−2		−.4		1.000		−.9

3. Divide the following:

(a) 16/.4 (b) 81/.009 (c) .125/5 (d) −2.525/.05
(e) 144/−.4 (f) −.49/−.007 (g) 100.58/−10 (h) −16.16/−4

4. Complete the following operations with fractions in the manner indicated:

(a) $^1/_2 + {}^3/_4 + {}^5/_3 + {}^5/_6 =$
(b) $2\,^2/_3 + {}^5/_8 + {}^7/_{12} =$
(c) $^7/_3 + 2\,^3/_4 - {}^2/_3 =$
(d) $^5/_3 + {}^6/_{15} - (-{}^9/_{15}) =$
(e) $^5/_6 - (-2\,^1/_6) - {}^7/_6 =$
(f) $(^6/_7)(^4/_5) =$
(g) $(^3/_4)(^5/_2)(-{}^2/_3) =$
(h) $(^3/_4) \div (^4/_3) =$
(i) $(^7/_8)(^4/_5) \div {}^3/_7 =$
(j) $6(-8 + 4) - {}^1/_2(-3 \div -6) =$
(k) $12[286 - (16)^2]\,[198 - (14)^2] =$
(l) $\dfrac{442}{11} - (-.426) =$
(m) $(^1/_2)(^2/_3)(0) =$
(n) $4^3 =$
(o) $(3)^4 - (5)^3 =$
(p) $a(ab) =$
(q) $(^2/_3)^2 =$
(r) $(x + y)^2 =$
(s) $12 - (-3)^2 =$
(t) $(fx)x =$
(u) $(x)(xy)(x^2y^2)(z) =$

5. Find the square root of each of the following:

(a) 3249
(b) 30625
(c) 306916
(d) 777.89
(e) 8876.9
(f) .00009
(g) .00025
(h) 9.0009
(i) $^1/_9$
(j) −.49
(k) 76453.678

6. Round each of the following to the nearest tenth:

(a) 14.36
(b) 24.32
(c) 17.798
(d) .098
(e) 1.011
(f) 90.25
(g) 87.95
(h) 56.35
(i) 48.575
(j) .125

7. How many significant digits are there in each of the following?

(a) 234.67
(b) 10.67
(c) .67
(d) .067
(e) 5004.009
(f) .009
(g) 7.2
(h) 0.01

3

FREQUENCY DISTRIBUTIONS AND GRAPHS

Often to make our data more meaningful and easier to handle we set up a frequency distribution and draw graphs of various kinds to represent it.

THE FREQUENCY DISTRIBUTION

Suppose that we have given a geography test yielding the score shown in Table 3.1. The first step is to determine the range. This is defined as the highest score minus the lowest score plus one. In Table 3.1 the highest score is 82 and the lowest is 28. In statistics we let a capital X stand for any raw score or any measurement. If we were working with two sets of data at the same time, we could use a capital Y for the other set. The range then is:

TABLE 3.1. Scores on a Geography Test

56	78	62	37	54	39	62	60
28	82	38	72	62	44	54	42
42	55	57	65	68	47	42	56
56	56	55	66	42	52	48	48
47	41	50	52	47	48	53	68

$$(X_H - X_L) + 1 \text{ or } (82 - 28) + 1 = 55$$

The second step is to decide how large each of the intervals in the frequency distribution is going to be. A widely accepted practice is to have between 10 and 20 intervals in any frequency table. When there are less than 10 intervals, the coarseness of the grouping makes for inaccuracies; when there are more than 20, the work becomes laborious. The size of the interval can be determined in a trial and error fashion as follows.

If we let each interval cover ten units, then the range will be included by 6 of these intervals, 55 ÷ 10 = 5.5 or 6. Ten then is too large as this results in only 6 intervals. Suppose that this time we try 5. Five goes into 55 exactly 11 times; hence 5 would be acceptable as the size of our interval. If we try 3, we find that 3 goes into 55 about 18 times; so 3 could be used for the size of the interval. Let us consider one more, 2. Two is contained in 55 about 28 times. We would reject this as the size of the interval as it results in too many intervals. Since we need between 10 and 20 intervals in our frequency distribution, we can take the average of 10 and 20, which is 15, and divide this into the range. In this case the range 55 divided by 15 results in a value between 3 and 4 and either of these could accordingly be used as the interval size (i).

If you observe frequency distributions made by others, you will note that there is a tendency to use odd numbers for the size of the class intervals. The advantage of this is that the midpoints of each of the class intervals will be whole numbers. In grouping data we find then that such numbers as 3 and 5 are commonly used. An exception to the above is the use of 10 which often makes a convenient interval size.

Suppose for the data in Table 3.1 we decide to use an interval with a size of 3. We would note this by writing that $i = 3$. We start building the frequency distribution that is shown in Table 3.2. The next problem is to decide where to start. A common practice is to let the bottom interval begin with a number which is a multiple of the class interval. In this case our lowest score is 28, the size of our interval is to be 3, so the bottom interval would begin at 27 and end at 29. These are the *integral limits*. Remember from the last chapter that in statistics we are going to deal with upper and

TABLE 3.2. Setting Up a Frequency Distribution

		f
81–83	/	1
78–80	/	1
75–77		0
72–74	/	1
69–71		0
66–68	///	3
63–65	/	1
60–62	////	4
57–59	/	1
54–56	𝍸///	8
51–53	///	3
48–50	////	4
45–47	///	3
42–44	𝍸	5
39–41	//	2
36–38	//	2
33–35		0
30–32		0
27–29	/	1
		$N = 40$

lower limits (or exact limits) and so this bottom interval actually begins at 26.5 and ends at 29.5. If we subtract the first of these from the latter, we obtain a value of 3 which is the size of the interval. This interval size is not apparent unless we are aware of the exact limits of the interval. This bottom interval also has a midpoint that is one-half of the distance between the lower and the upper limits. The midpoint for this interval then is 26.5 + 1.5 which equals 28. As noted above, when the size of an interval is an odd number, the midpoint turns out to be a whole number.

After deciding upon the bottom interval we continue to set up the rest of the intervals by increasing each one by 3. We stop when we reach the interval 81–83 which contains the highest score in the distribution. The usual practice is to set up a frequency distribution with the lowest scores at the bottom.

The next task is to tally the scores. We take the scores one at a time and enter them to the right of the appropriate interval by making a tally mark. When this is finished we combine the tallies in the column to the right and we head this column with *f* which stands for frequencies. This column is summed, and we write the total at the bottom. We can write a capital *N* in front of this sum to denote that the number of cases is 40 or we can write $\Sigma = 40$. This last symbol which is the Greek capital sigma is read as "summation of" or "sum of." It is one of our commonest statistical symbols. You should note that for intervals where there were no tallies, we entered a 0 in the *f* column. Data that have been set up in such a frequency distribution are said to be grouped, and formulas used with them are applicable to this type of data only. As opposed to this, we have ungrouped data, and the formulas applied to them are referred to as raw score formulas.

Suppose that we again set up the same geography test scores in a frequency table. This time the size of the interval will be 5. Since our lowest score is 28, the lowest interval will be 25–29. This interval has exact limits of 24.5 and 29.5 and a midpoint of 27. This second table appears in Table 3.3. This time we have only 12 class intervals instead of 19 which resulted when

TABLE 3.3. Setting Up a Frequency Distribution

		f
80–84	/	1
75–79	/	1
70–74	/	1
65–69	////	4
60–64	////	4
55–59	𝍸 //	7
50–54	𝍸 /	6
45–49	𝍸 /	6
40–44	𝍸 /	6
35–39	///	3
30–34		0
25–29	/	1
		$\Sigma f = 40$

we used an interval size of 3. In the actual working of these data, the interval of 5 would be preferred.

At this point we summarize the steps which we go through in setting up a frequency distribution or frequency table.

1. Determine the range.
2. Divide this by 15 to give an idea of the approximate size of the class interval.
3. Set up the frequency distribution, beginning at the bottom. Let the lowest interval begin with a number which is a multiple of the class interval.
4. Tally the frequencies.
5. Summarize these under a column labeled *f*.
6. Add this column and record the number of cases at the bottom. It might be mentioned here that just because this number obtained by adding the frequencies is the same as the number of cases, it does not follow that no mistake has been made. To check the work, another distribution should be set up and the scores again tallied.

GRAPHS

The Frequency Polygon. Of all the graphic devices used in illustrating statistical materials, probably the one most frequently encountered is the frequency polygon. The reasons for this are that it is very easy to construct and it is basically very simple and easy to interpret. Several of these, using different types of lines for their portrayal, may be placed on the same axes.

Construction of the Frequency Polygon. In all work with graphs, two axes are used. The vertical one is always labeled the Y-axis and values taken along this axis are called ordinates. The other axis, the X-axis, called the abscissa, is horizontal and meets the Y-axis at right angles at a point called the origin (O). In constructing graphs of the frequency polygon type, we have the X-axis longer than the Y-axis. Usually a ratio of three to two or four to three will result in a good graph. For example, if the X-axis is 6 inches in length, the Y-axis should be about 4 inches. Or if another graph had an X-axis of 8 inches, then the Y-axis should be about 6 inches.

To make graphs that can be easily and exactly read, a good type of graph paper should be obtained. The type that has ten squares to the inch is the best for statistical work. Also, it is easier to build graphs of the desirable proportions if the X-axis is placed on the wider wide of the graph paper. In building a frequency polygon, we always place the frequency values on the Y-axis. A small f is placed along the Y-axis. The scores go on the X-axis. To illustrate this suppose that we construct a frequency polygon for the data in Table 3.3. The highest frequency is 7, so we need no more than 7 units on the Y-axis of Figure 3.1. On the X-axis we have placed score values.

The next step is to plot the frequencies. For any one of the class intervals in Table 3.3 we can make an assumption that if all the scores in that interval were averaged, this average would be equal to the midpoint of the interval. Then it follows that the midpoint is the value which best represents any given interval. In plotting our frequencies we place the values above the corresponding midpoints of each of the class intervals. Let us take the first or bottom interval of our table, 25–29. This interval has a midpoint of 27 and a frequency of 1. We go along the X-axis until we come to 27 and then we go up one unit above this point and place a point there. The midpoint of the next interval is 32 and this interval has a frequency of 0. So at 32 on the next axis, a mark is placed at 32. Continuing, the midpoint of the third interval is 37 with a frequency of 3. So at a point 3 units above 37 on the X-axis we place our next point. We proceed like this until all of the frequencies have been plotted. Then we take a ruler and connect these points with straight lines.

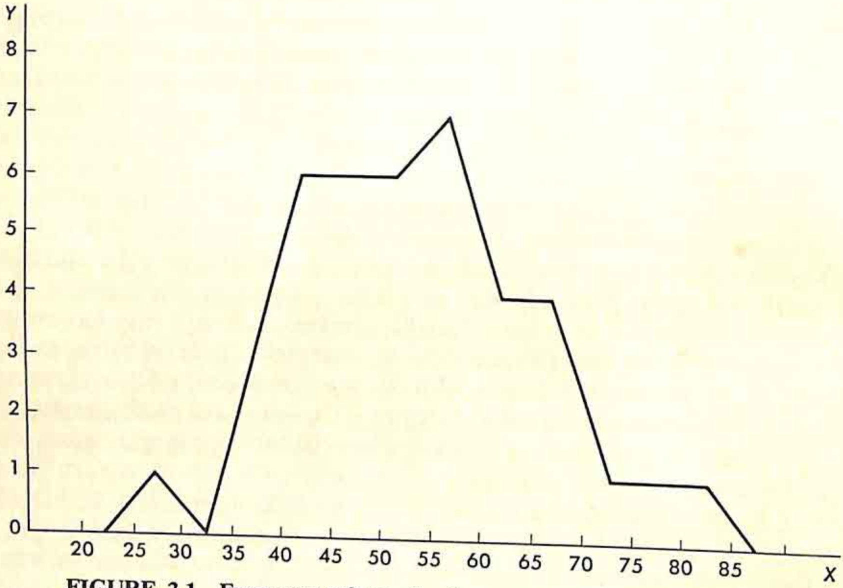

FIGURE 3.1 **Frequency polygon for the geography scores in Table 3.1.**

Rather than leave our graph suspended in space, we assume that there is another interval above and below those shown in our table and that each of these has a frequency of 0. So at 22, the midpoint of the bottom one of these we place a check mark on the X-axis, and a similar mark is placed at 87 to take care of our top. The graph is now anchored at both ends. Some find it much more convenient to set up their frequency polygons with the numerical values of the midpoints of the intervals on the X-axis rather than the score values shown in Figure 3.1. The resulting picture is the same, no matter which method is used in the construction.

As mentioned earlier, the frequency polygon is very useful in picturing two or more distributions at once. In Figure 3.2 is shown the distribution of the intelligence quotients of three groups of kindergarten children in three different schools of a city. From such a graph, similarities and differences are very apparent.

Constructing Frequency Polygons When Frequencies Differ. In many cases, as in Table 3.4, we find that the frequencies in the different groups we wish to plot differ considerably. In this table we have the distribution of intelligence test scores for 180 children in one school (f_1) and of 50 children in another school (f_2). A little consideration will show that if we attempt to plot both of these distributions on the same axes as they are now set up, we shall get into trouble. Because of the different size frequencies in the different intervals, it may be impossible to plot both on the same graph or one graph will end up being so far above the other that a direct comparison is impossible.

TABLE 3.4. Intelligence Test Scores of Children in Two Schools

(1)	(2) School A f_1	(3) School B f_2	(4) P_1	(5) P_2
150–159	1	1	.6	2
140–149	4	2	2.2	4
130–139	8	4	4.4	8
120–129	12	17	6.7	34
110–119	31	18	17.2	36
100–109	69	6	38.3	12
90–99	32	2	17.8	4
80–89	18	0	10.0	
70–79	4	0	2.2	
60–69	1	0	.6	
	Σ = 180	50	100.0%	100%

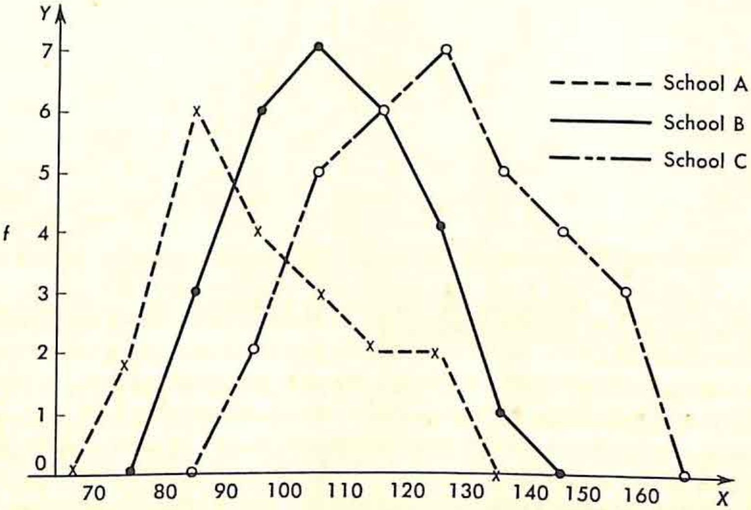

FIGURE 3.2 Distribution of Stanford-Binet IQ's for groups of kindergarten children in three schools of a city.

In cases like these, the best practice is to convert each frequency into a percentage or proportion and to plot these. We have converted the data in

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Table 3.4 to percentages. Notice now that we have two more columns, P_1 and P_2, the percentages of each group in the various intervals. We compute such percentages by dividing the number in each interval by the number in the total group and multiplying this result by 100. Taking the bottom frequency in the f_1 column, we have $^1/_{180}(100)$ which results in .6 percent. In this manner we compute all of the percentages.

But an easier way to change a series of numbers to percentages is to find a constant multiplier and then to multiply each frequency in the distribution by this constant multiplier. Let us first consider the f_1 distribution in Table 3.4. There are 180 cases. It follows then that each case is equal to $^1/_{180}(100)$ percent, which in this case is .555. This .555 then is our constant multiplier. Multiplying each frequency in column (2) by this constant and rounding each results in the values shown in column (4). A check on the work is that the sum of this latter colum is 100 percent. Sometimes there are minor variations in this sum brought about by rounding operations. The f_2 data in column (3) are taken next. This time the constant multiplier is $^1/_{50}(100)$ which equals 2. So we multiply each of these frequencies by 2 and get the results shown in column (5).

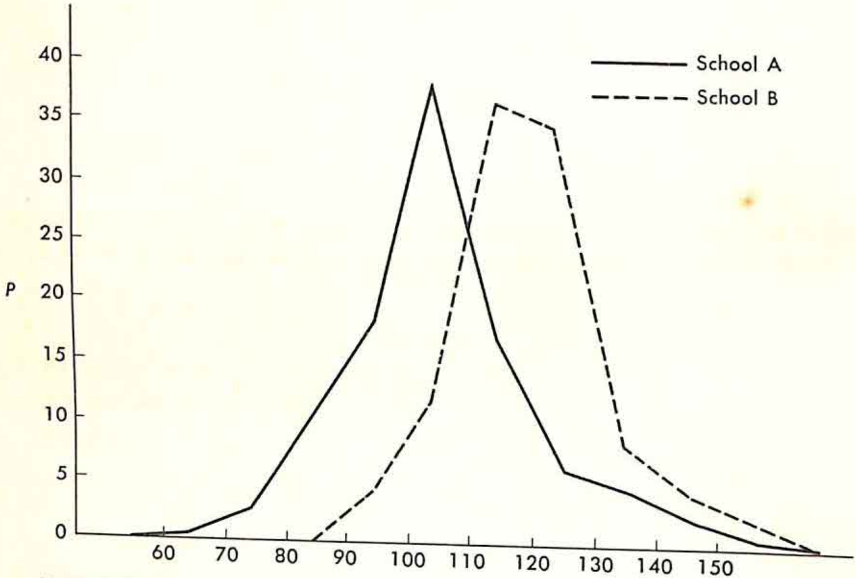


FIGURE 3.3 Distribution of scores on an intelligence test in two schools with the frequencies converted to percentages.

The next step is to set up the frequency polygon in the same fashion as described earlier. This time, though, we place our percentages, instead of the frequencies, on the Y-axis. The largest percentage in columns (4) and (5) is under 40 (38.3); hence the largest value needed on our Y-axis is 40. Along the X-axis we have the test scores plotted in the usual manner (see Figure 3.3). We plot our percentages above the midpoints of the appropriate intervals on the X-axis and connect these points with straight lines. These lines should be constructed using different types of lines or different colors. Since we have two graphs on the same axis, we include a legend showing which polygon represents which set of data.

Smoothing Curves. After one has constructed frequency polygons for classroom groups with rather small frequencies, one is usually impressed with the ragged shapes of the resulting graphs. The question naturally arises as to whether or not the picture would be the same if we had a larger sample upon which we could base our graphs. Many of these irregularities are what we describe as being chance fluctuations, and as the number of cases is increased in any sample, these chance fluctuations tend to become smoother. We have a method called the *Method of Running Averages* of smoothing a curve which we use to compensate for small frequencies.

To illustrate this process we have again reproduced our geography test scores in Table 3.5. Note that we have added two more intervals to our table, one at the top and another at the bottom, each with a 0 frequency to bring the curve down to the X-axis. Let us start the rounding.

TABLE 3.5. Smoothing a Curve

	f	f_s
85–90	0	.3
80–84	1	.7
75–79	1	1
70–74	1	2
65–69	4	3
60–64	4	5
55–59	7	5.7
50–54	6	6.3
45–49	6	6
40–44	6	5
35–39	3	3
30–34	0	1.3
25–29	1	.3
20–24	0	.3

We take the top interval with a frequency of zero. If there were another interval above this, its frequency would also be zero. The frequency of the interval below 80–84 is equal to 1. The average of these three frequencies is then $\frac{0 + 0 + 1}{3} = .3$, the smoothed frequency for the interval 85–90. Next we move down to the interval 80–84. The frequency for the interval above this one is 0, its frequency is 1, and the frequency of the interval below it is 1. The smoothed frequency for this interval is then $\frac{0 + 1 + 1}{3} = .7$. The smoothed frequency for the next interval, 75–79, then becomes $\frac{1 + 1 + 1}{3} = 1$. In this manner we continue until we have the smoothed frequencies for the entire table and the column labeled f_s is completed.

The next task is to plot this new set of frequencies. We set up the frequency polygon in the same manner as previously and plot the smoothed frequencies above the midpoints of their representive intervals, as shown in Figure 3.4. This time, instead of connecting our points with a ruler, we draw a freehand curve which goes through most of the points. If some of the points are still out of the line of the curve, we do not attempt to go through them. By doing this we continue the smoothing process.

Several things should be noted here. This smoothed curve may be used as a picture of what the distribution would be like if we had more cases. Of course, possibly we are wrong. The irregularities that we had in the sample may be the true state of things and the smoothing may have brought about error. The chances of this, however, are small. We should further emphasize here that this smoothing process of curves is not normalizing. We shall take this up later in another chapter.

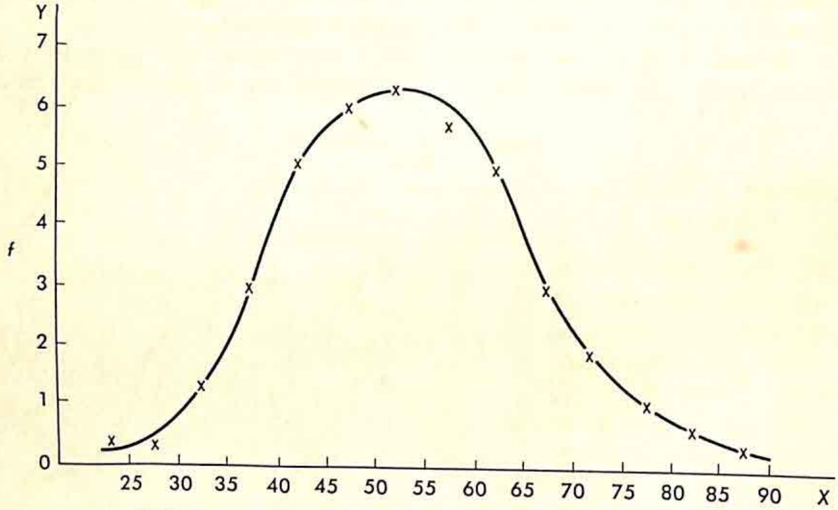


FIGURE 3.4 **Smoothed curve for the data in Table 3.4.**

TYPES OF CURVES

After making a few frequency polygons, one soon notices that the curves tend to have shapes that could be classified into types. Sometimes the frequencies tend to pile up on the left-hand side of the graph with a tail extending to the right, as shown in Figure 3.5. Such a curve is said to be skewed. If the tail goes to the right, we label this type of skew as being positive. (This conforms with the mathematicians' practice of calling the right end of a line positive and the left end negative.) Figure 3.6 shows the opposite of the above, and this condition is called negative skewness. The student should remember that it is the tail of the distribution which determines the sign of the skewness and not the location of the pileup of scores.

Another type of curve that is by no means rare is the bimodal curve shown in Figure 3.7. This curve has two peaks or modes. It is also possible for a curve to have more than two modes.

The last type of curve that we shall consider is the one referred to as the normal or bell-shaped curve. This is the picture of the so-called normal distribution. A good share of the statistical operations covered in this book are based upon the assumption that the data being handled are normally distributed. Since this type of curve is of such major importance, considerable time and space will be devoted to it later, at which time we shall discuss its properties and uses.

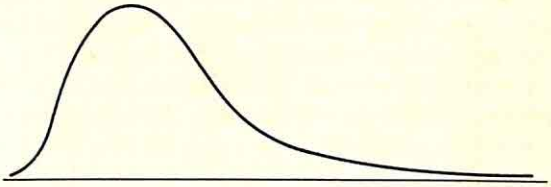


FIGURE 3.5 A positively skewed curve.

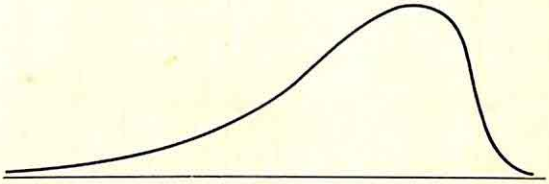


FIGURE 3.6 A negatively skewed curve.

OTHER TYPES OF GRAPHS

The Histogram. The histogram (Figure 3.9) is very similar in its construction to the frequency polygon. Everything is the same up to the point of plotting the frequencies. The histogram in Figure 3.9 is again based upon data in Table 3.2. Here is how this graph is made. The bottom interval 25–29 has a frequency of 1. Starting at the lower limit of this interval, 24.5, the interval is marked off and, since the interval has a frequency of 1, this interval has a height of 1 on the *Y*-axis. The next interval has a frequency of 0; hence there is a gap in our histogram at this point. At the interval beginning with 34.5 we mark off a bar with a height of 3, since there are three scores in this interval. We continue this process until we have entered all the frequencies into the graph. If the midpoints of the top line of each of the columns comprising the histogram were connected, one would have the same frequency polygon previously constructed.

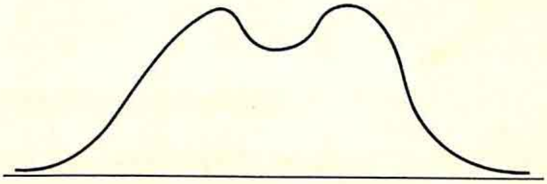


FIGURE 3.7 A bimodal curve.

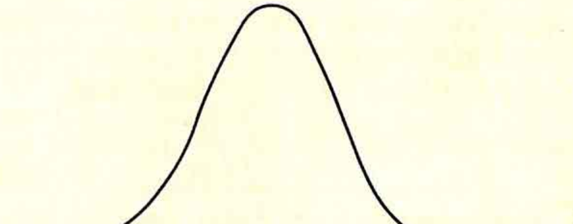


FIGURE 3.8 The normal or bell-shaped curve.

The histogram is more time-consuming to make than the frequency polygon. Also, only one can be clearly placed on any set of axes unless the second is reversed and plotted down from the X-axes. We saw how several frequency polygons were placed on the same axes to make comparisons and illustrate differences. This cannot be done easily and clearly with histograms. Their use is thus more limited than that of the other type of graphs.

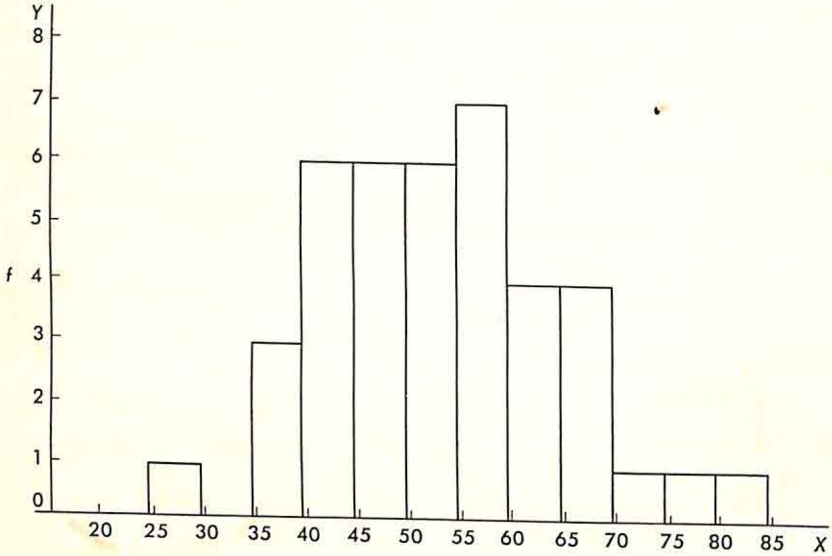


FIGURE 3.9 Histogram for the geography scores in Table 3.1.

Graphs for Frequencies, Proportions, and Percentages. In Table 3.6 are figures which give the number of drop-outs in five colleges of two universities

TABLE 3.6. Drop-outs in the Various Colleges of Two Universities

	f University A	f University B	p University A	p University B	P University A	P University B
Engineering	440	640	.48	.58	48%	58%
Liberal Arts	220	220	.24	.20	24%	20%
Agriculture	120	120	.13	.11	13%	11%
Home Economics	80	80	.09	.07	9%	7%
Fine Arts	60	40	.06	.04	6%	4%
	920	1100	1.00	1.00	100%	100%

over a period of two years. The frequencies have been converted into both proportions and percentages, p's and P's, respectively. Many types of graphs can be used to display pictorially these data. One of the commonest of these is the bar graph (Figure 3.10.). Another one frequently encountered is the pie diagram shown in Figure 3.11. The weekly news magazines, the business sections of the daily newspapers, and our own professional journals include many illustrations of the various types of these graphs. There is neither reason nor time to consider any more of these in this book.

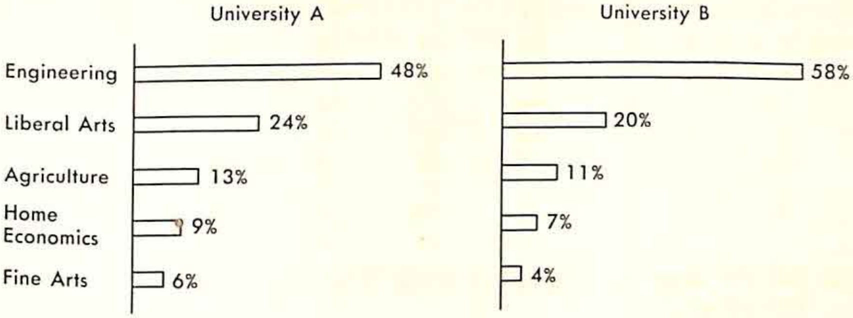


FIGURE 3.10 Bar graphs for the data in Table 3.5.

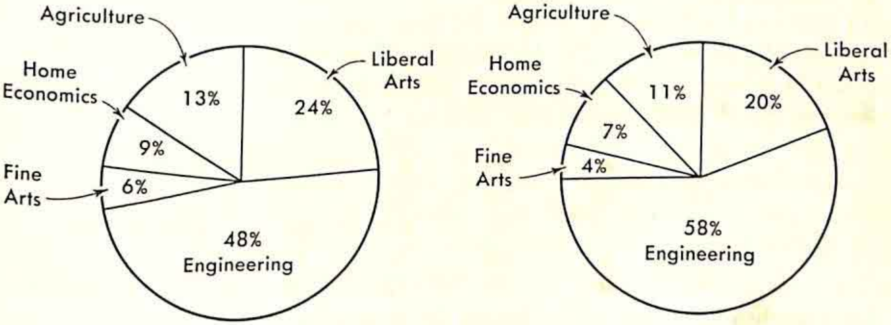


FIGURE 3.11 Pie diagrams for the data in Table 3.5.

Exercises

1. What are the exact limits and interval sizes for each of the following intervals?

(a) 0–4
(b) (−3)–(−8)
(c) 2–4
(d) (−2)–(+4)
(e) 98–103
(f) 1.5–3.5
(g) 5–10
(h) .50 − .75

2. What class intervals would you use in working with the following distributions?

	N	*Low Score*	*High Score*
(a)	250	0	98
(b)	250	−1	+1
(c)	30	1	23
(d)	20	0	15

3. The following scores were made on a 55-item test of elementary statistics:

53	47	42	39	36
52	46	41	39	35
52	45	41	39	35
52	44	41	39	34
52	44	40	39	32
51	44	40	38	31
50	43	40	38	31
50	43	40	38	30
49	43	40	37	28
48	42	39	37	22
47	42	39	36	

(a) For the above data set up a frequency distribution starting with a bottom interval of 21–23.
(b) Make another distribution, this time starting with an interval of 20–22.
(c) Compare the two distributions.
(d) Construct a frequency polygon for these scores.
(e) Smooth the data and plot the smoothed frequencies on the same axes.

4. A third grade class of 36 pupils is given 100 words to spell. Their scores represent the number of words spelled correctly.

56	70	21	15	75	88
66	87	33	82	43	74
43	95	78	64	72	36
45	27	52	87	84	92
38	33	60	73	71	74
40	67	78	82	100	53

(a) Establish a frequency distribution for these data.
(b) Construct a frequency polygon for these data.
(c) Construct a smoothed frequency polygon on the same axes.
(d) Is the distribution skewed positively or negatively?
(e) Construct a histogram for these data on the same axes.

5. The distribution of total mental age scores of two groups of high school seniors made on the *California Test of Mental Maturity* follows:

	School A	*School B*
	f	*f*
250–259	1	0
240–249	8	0
230–239	12	4
220–229	24	20
210–219	32	32
200–209	12	48
190–199	8	44
180–189	0	12
170–179	1	12
160–169	1	8
150–159	0	8
140–149	1	2
130–139	0	2

(a) Plot these two distributions on the same axes.
(b) Describe each of the distributions.

4

AVERAGES

One of the most important and basic things to do with a group of scores is to compute the average. This average gives us an indication of the way in which the scores pile up at the middle of a distribution. Usually three averages are considered: the mean, the median, and the mode. Of these three measures of central tendency, the mean is the one most commonly encountered, and, as we shall see later, it is this one that is basic to many other statistical computations. In these averages we have a single score which represents a group of individuals.

THE ARITHMETIC MEAN

The mean is another term for arithmetic average. Everyone who has ever computed an average has computed a mean. Very simply, the mean is the sum of the scores divided by the number of cases. Suppose that we have five scores; $X_1 = 10$, $X_2 = 12$, $X_3 = 15$, $X_4 = 18$, and $X_5 = 20$. Then the mean ($\bar{X}$), read as bar X, would be:

$$\begin{aligned}\bar{X} &= \frac{X_1 + X_2 + X_3 + X_4 + X_5}{N} \\ &= \frac{10 + 12 + 15 + 18 + 20}{5} \\ &= \frac{75}{5} = 15\end{aligned}$$

Instead of writing the equation for the mean as above, we shorten it like this:

$$\bar{X} = \frac{\Sigma X}{N} \tag{4.1}$$

where

$\bar{X}$ = mean.
ΣX = the sum of the scores. Σ is the capital letter for S in Greek and is customarily used as an abbreviation for sum.
N = the number of cases.

The Mean for Grouped Data. When the scores have been entered into a frequency table, the mean is computed as is shown in Table 4.1. In this table, the same scores which appear in Table 3.1 are again used. The method illustrated here is that of an arbitrary reference point.[1] The steps that we go through in finding the mean are outlined below:

TABLE 4.1. Computation of the Mean for Grouped Data

	(1) f	(2) x'	(3) fx'
80–84	1	6	6
75–79	1	5	5
70–74	1	4	4
65–69	4	3	12
60–64	4	2	8
55–59	7	1	7
50–54	6	0	0
45–49	6	−1	−6
40–44	6	−2	−12
35–39	3	−3	−9
30–34	0	−4	0
25–29	1	−5	−5
	$N = 40$		$\Sigma = 10$

1. The frequency table is set up as described in Chapter 3.
2. We take the midpoint of one of the intervals as the arbitrary reference point. As far as the result is concerned, it makes no difference where we start. In the example illustrated, the midpoint of the interval 50–54, 52 is taken as the arbitrary reference point.
3. A second column is set up. This time it is labeled x' and can be read as deviations from the arbitrary reference point. Since the interval 50–54 was taken as the interval containing our reference point, there is no deviation and a 0 is placed in column (2) for this interval. The interval of 55–59 deviates one interval from the arbitrary reference point and a 1 is entered for this interval in column (2). This is continued upward until each interval has a value. Going back to our selected reference point, we do the same thing for the intervals below the mean. However, this time we place a minus sign in front of each of our deviations.
4. Next we multiply each f by its x' and enter the product in the third column labeled fx'.
5. Sum this column.
6. The mean is then computed by substituting in this equation:

$$\bar{X} = M' + \frac{\Sigma fx'}{N}(i) \tag{4.2}$$

---

[1] See Chapter 5, p. 52, for an explanation of this arbitrary reference point, which is referred to as the "assumed mean" by some writers.

where

M' = arbitrary reference point.
i = the size of the class interval and the other symbols are as previously defined.

Then for our problem:

$$
\begin{aligned}
\bar{X} &= 52 + \frac{10}{40}(5) \\
&= 52 + \frac{50}{40} \\
&= 52 + 1.25 \\
&= 53.2
\end{aligned}
$$

Suppose that we work the problem over again, this time selecting the midpoint of another interval as the arbitrary reference point. It was noted above that it made no difference where we started. This time the data are set up in Table 4.2 and the computational work is recorded beneath it. We begin by selecting the midpoint of the bottom interval as our reference point and placing a 0 for this interval for the x' column. The next interval is given a value of 1, and this is continued up to the top interval which receives a value of 11. The fx' value is then computed for each interval and these are summed. Substitution into formula 4.2 results in a mean of 53.2 which is identical with the value obtained in the first solution. We could repeat this

TABLE 4.2. Computation of the Mean and Median for Grouped Data

	(1) f		(2) x'	(3) fx'
80–84	1		11	11
75–79	1		10	10
70–74	1		9	9
65–69	4		8	32
60–64	4		7	28
55–59	7	18	6	42
50–54	6		5	30
45–49	6	16	4	24
40–44	6		3	18
35–39	3		2	6
30–34	0		1	0
25–29	1		0	0
	$N = 40$			$\Sigma = 210$

$$
\begin{aligned}
\bar{X} &= M' + \frac{\Sigma fx'}{N}(i) \\
&= 27 + \frac{210}{40}(5) \\
&= 27 + 26.25 \\
&= 53.2
\end{aligned}
$$

process ten more times, that is, we could use the midpoint of each of the other intervals as arbitrary reference points. If our work is done correctly, each solution will result in a mean of 53.2. When we started with the bottom interval, all of our values were positive, but our values became rather large. Some prefer this to the method used first. However, it is felt that if no calculating machine is available, it is more practical to begin by selecting an interval near the center of the distribution.

It might be noted that the mean obtained here is not identical with that obtained by adding all of the scores and dividing by N. Such differences between the means produced by the two methods are usually very small and of no practical significance. We say that this difference is brought about by the error of grouping. This concept will be discussed in detail in the next chapter.

THE MEDIAN

The median (*Mdn*) is defined as that point in the distribution which has 50 percent of the cases on each side of it. In other words, it is the midpoint of the distribution.

Finding the Median for Ungrouped Data. When the number of scores is small, the easiest way to obtain the median is to arrange the scores from high to low, count the number of scores, divide this number by 2, and count up (or down) to find that point which has this number of scores below it. Suppose that we have the following 11 scores:

20, 19, 18, 17, 16, 15, 14, 13, 12, 11, 10

In this case 15 has five scores on each side of it and, hence, 15 is the median.

Here is another series:

20, 19, 18, 17, 16, 15, 14, 13, 12, 11

In this series we count up 5 cases (or down 5) and this leads us to the point 15.5 which is the median.

The above two series are easy ones and the medians can be obtained by inspection. Suppose that our series is like this:

21, 20, 19, 18, 17, 17, 17, 16, 15, 14

This time we again have 10 cases and we wish to find a point which has five scores on each side of it, that is, N divided by 2. We start up from the bottom counting cases. The 14, 15, and 16 occupy the first three positions. However, the three 17's make it impossible to finish this problem as we did the first two. We can continue up to the bottom of the number 17 or to the top of 16. This point is 16.5. Here we have to interpolate. We have used up 3 cases and we need 2 more. These three 17's are assumed to be spread equally through the interval 16.5–17.5, and, since we want only two of these 17's, we shall take a point which is two-thirds of the way through this interval as the median. In summary,

$$\text{Mdn} = 16.5 + \frac{2}{3} = 16.5 + .67 = 17.2$$

Let us repeat this for another series of numbers.

20, 19, 16, 16, 15, 15, 15, 14, 10, 8, 7

This time there are 11 numbers in our series and 11 divided by 2 is 5.5. So we are looking for a point with 5.5 cases on each side of it. We start from the bottom counting cases and this time, after counting over 4 cases, we find that we are at the bottom of the number 15 and we again have to interpolate. Then the median is obtained as follows:

$$\text{Mdn} = 14.5 + \frac{1.5}{3} = 14.5 + .5 = 15$$

Finding the Median for Grouped Data. To illustrate this technique, suppose that we use the data in Table 4.2. The first step again is to take 50 percent of N or to divide N by 2. In this case $N/2$ is 20. We start at the bottom of the frequency distribution and begin counting up 20 cases. The upper limit of the interval 45–49 has 16 cases below it. If we add to this the frequency of the next interval, which is 6, we will have 22 cases, which is more than our needed 20. We go back to the interval 45–49 and interpolate to find the median. The interval 50–54 contains 6 cases, 4 of which we need. We assume that these 6 cases are spread equally through the interval and since we need 4 of these, we must go $^4/_6$ of the distance through the interval, the size of which, for these data, is 5. Hence we add $^4/_6$ or $^2/_3$ of 5 to the upper limit of the interval 45–49 or to the lower limit of the interval 50–54, which is the same for both of these, 49.5.

This may be summarized as below:

$$\begin{aligned}\text{Mdn} &= 49.5 + \frac{4}{6}(5)\\ &= 49.5 + \frac{20}{6}\\ &= 49.5 + 3.3\\ &= 52.8\end{aligned}$$

The computation of the median may always be checked by repeating the process illustrated above and beginning at the top and counting down $N/2$ cases. For the data in Table 4.2, we find that the lower limit of the interval 55–59 has 18 cases above it. At this point we interpolate as follows:

$$\begin{aligned}\text{Mdn} &= 54.5 - \frac{2}{6}(5)\\ &= 54.5 - \frac{10}{6}\\ &= 54.5 - 1.7\\ &= 52.8\end{aligned}$$

which is the same result we obtained before. Notice this time that a minus sign appears in the operation. This is necessary because we wish to continue going down.

THE MODE

A third average which we shall mention very briefly here is the mode, the symbol of which is *Mo*. For ungrouped data, the mode is defined as that measure which occurs most frequently. When the data have been arranged into a frequency table, the mode is defined as the midpoint of the interval containing the largest number of cases. For example, the mode of the data in Table 4.2 is 57, the midpoint of the interval 55–59.

WHEN TO USE THE DIFFERENT AVERAGES

Of the three measures of central tendency, the mean is the one most frequently encountered. However, there are situations where its use is not justified. Suppose that ten individuals contributed the following to an organization:

$20.00
.25
.25
.25
.15
.10
.10
.10
.05
.05
$21.30

The mean for these figures is $2.13, whereas the median is 12 cents. The latter is a much more accurate picture of the average contribution than is the former. It should follow from this that when there are extremely high or low scores, the median should be used in preference to the mean. The mean is to be used when any distribution tends to be normal. When a distribution departs from normal and is badly skewed, the mean is pulled toward the tail and is always higher (or lower) than the median and mean, as is shown in Figures 4.1 and 4.2. The three measures of central tendency are the same when the distribution is normal. The student should note here that every score in a distribution has an effect upon the mean. In the above example,

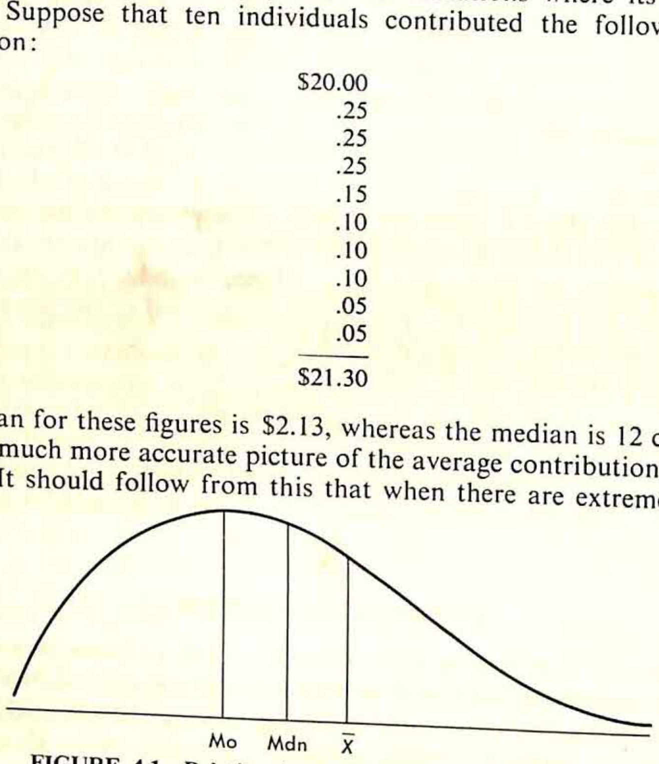


FIGURE 4.1 Relative sizes of the different measures of central tendency in a positively skewed distribution.

the top contribution could have been 2 million dollars and the median would still be 12 cents, but the mean would have increased considerably if the top value had been so changed.

The mean is used when more computations are to be made later. The student will see that many of our other statistics are determined from the mean. On the other hand, the median has very limited use in statistical work. The median does have the advantage of being easy and rapid to compute. Of these two measures, the mean is the more reliable. By this we mean that from sample to sample it tends to vary less. This will be discussed later when we study sampling statistics.

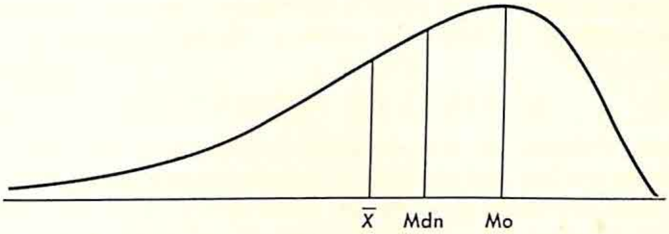


FIGURE 4.2 Relative sizes of the different measures of central tendency in a negatively skewed distribution.

The third measure, the mode, is seldom used. It is very easy to compute, but it suffers from the fact that it is very unreliable. Let us refer to Table 4.2 again. We noted that the mode for this distribution was 57, the midpoint of the interval 55–59. Now suppose one of these frequencies is removed from this interval and placed in the interval 45–49. This interval would now have a frequency of 7 and the mode of the distribution would now be 47. Then a change in the location of just one case brought about a change in the mode of 10. A statistic which fluctuates so is too unreliable for any but the crudest of uses.

AVERAGING MEANS

Frequently we have two or more samples where the mean of each sample has already been computed and we wish to find the mean of all of the individuals taken together in one group. We do this by computing the so-called weighted mean. Suppose that as a result of giving a test to three different groups, we obtain the following results:

$$\bar{X}_A = 70 \qquad N_A = 30$$
$$\bar{X}_B = 60 \qquad N_B = 60$$
$$\bar{X}_C = 80 \qquad N_C = 10$$

where $\bar{X}$ and N stand for the means and the number of individuals in groups A, B, and C. To obtain the mean of the total, we multiply each mean by the number of individuals in its group, add these, and then divide this sum by the total number of cases, N_T.

$$\begin{array}{ccccccc} \bar{X} & & N & & \\ 70 & \times & 30 & = & 2100 \\ 60 & \times & 60 & = & 3600 \\ 80 & \times & 10 & = & 800 \\ & \Sigma = & 100 & \Sigma = & 6500 \end{array}$$

$$\bar{X}_T = \frac{6500}{100} = 65$$

It will be noted that this mean of 65 is different from that obtained by averaging the three means, which in this case would be 70. Only when the number of individuals in each sample is the same can the weighted or grand mean be obtained by averaging the means of all the samples.

CENTILES OR PERCENTILES

While these statistics are not averages, they are very similar in their computation to the median and are therefore included in this chapter. A centile or centile point is defined as a specific point in a distribution which has the percent of cases below it which corresponds to the value of the centile. For example, the 88th centile (C_{88}) is that point in the distribution which has 88 percent of the cases below it and C_{18} is that point which has 18 percent of the cases below it. The median is then the 50th centile.

All of the other centile points may be computed in the same fashion as the median. To illustrate how this goes we shall use the data which appear in Table 4.3. In computing the median the first step was to take 50 percent of N (or to divide N by 2). Each of the other computations starts out in the same way. Suppose that we wish to find C_{18} for the data in Table 4.3. The

TABLE 4.3. Computation of the Various Centile Points

	f	
130–139	2	
120–129	6	
110–119	8	16
100–109	12	
90–99	20	
80–89	34	
70–79	43	
60–69	33	
50–59	22	
40–49	11	20
30–39	7	
20–29	2	
	$N = 200$	

first step is to take 18 percent of N or 18 percent of 200 which is 36. We begin to count up from the bottom to find which point has 36 cases below it. We proceed up to the lower limit of the interval 50–59. This point has 20 cases below it. At this point we have to interpolate, as we now need 16 of the cases in the next interval. This is done as follows:

$$
\begin{aligned}
C_{18} &= 49.5 + \frac{16}{22}(10) \\
&= 49.5 + \frac{160}{22} \\
&= 49.5 + 7.3 \\
&= 56.8
\end{aligned}
$$

Next we shall compute the 88th centile, C_{88}. We could start out by taking 88 percent of N and counting up from the bottom as we did for C_{18}. However, we can make our work much easier, by taking 12 percent of N and by counting down from the top. For example,

$$
\begin{aligned}
12\% \text{ of } N &= 12\% \text{ of } 200 = 24 \\
C_{88} &= 109.5 - \frac{8}{12}(10) \\
&= 109.5 - \frac{80}{12} \\
&= 109.5 - 6.7 \\
&= 102.8
\end{aligned}
$$

Several of the centile points have special names, such as median for C_{50}. The centile point C_{25} is known as the first quartile and is written as Q_1. Similarly, C_{75} is known as the third quartile and is written as Q_3. In addition to these there are nine decile points which divide the distribution into ten equal parts. C_{10} would be the equivalent of the first decile, C_{20} the second, etc.

Exercises

1. The following figures represent the scores of 57 university students on a test in educational measurement.

	f
57–59	1
54–56	1
51–53	5
48–50	9
45–47	5
42–44	8
39–41	10
36–38	6
33–35	4
30–32	7
27–29	0
24–26	1

Compute the three averages for these data.

2. The following distribution of scores was made on a clerical-sales scale of an interest inventory.

Score	*f*
15	1
14	10
13	10
12	16
11	20
10	15
9	17
8	20
7	14
6	18
5	19
4	15
3	3
	178

   (a) Compute the three measures of average for these data.
   (b) Evaluate each as to applicability to these data.

3. The following scores were obtained by a group of university students on the *Otis Self-Administering Test of Mental Ability*.

71	61	54	50
70	60	54	50
69	59	54	49
69	58	54	47
69	58	53	40
64	57	52	39
64	56	52	34
63	55	51	30

   (a) Group these data and find the mean.
   (b) Draw a frequency polygon for these scores.
   (c) What can you say about the usefulness of this test with this group?

4. Two junior civics classes take the same midterm test. Their scores are:

	Class I	*Class II*
71–75	1	0
66–70	0	1
61–65	2	3
56–60	4	3
51–55	3	4
46–50	7	6
41–45	6	5
36–40	4	5
31–35	5	4
26–30	2	3
21–25	1	0
16–20	3	4
11–15	0	1
6–10	1	0
1– 5	0	1
	$N = 39$	$N = 40$

(a) What is the mean, median, and mode of each class?
(b) What is the mean of the total group ($N = 79$)?
(c) What is the 25th centile of each group?

5. The *Wechsler Intelligence Scale for Children* is given to a class of 24 fourth grade pupils. Their IQ's are:

98	115	122	99
111	99	113	101
108	103	95	89
100	101	104	107
96	114	116	113
103	90	100	102

(a) What is the mean IQ of the class?
(b) What is the median IQ?
(c) Can you tell if the scores are normally distributed, knowing the mean and the median?
(d) What is the mode of this distribution?
(e) Find the following centile points in this distribution:

10th	25th	33rd
75th	80th	95th

6. The following are scores of small groups on a series of psychological tests. For each, compute the mean, median, and mode.

(a) 24, 26, 27, 29, 30, 31, 32, 33, 34.
(b) 12, 22, 26, 27, 28, 30, 30, 31.
(c) 8, 9, 10, 10, 10, 11, 11, 12, 12, 13, 18.
(d) 12, 14, 14, 14, 14, 16, 18, 20.
(e) 20, 21, 22, 23, 25, 26, 28, 30.
(f) 2, 4, 5, 6, 6, 6, 7, 7, 8, 12.

7. One class of 33 students had a mean score of 55 on a statistics test. Another class of 22 had a mean score of 60 on the same test. Compute the mean of the two groups combined for this test.

5

VARIABILITY

In the previous chapter we were concerned with the computation, meaning, and use of averages. A little thought will soon point up the fact that averages in themselves do not adequately describe a distribution. Averages locate the center of a distribution but tell us nothing as to how the scores or measurements are arranged in relation to the center. Let us take two illustrations. Suppose that we have two distributions of scores on the same test, each with a mean of 67. In the first of these the highest score is 72 and the lowest is 62. The second distribution has a high score of 107 and a low score of 25. The range of the first distribution is 11 and that of the second distribution is 83. To give a much better picture of a distribution it follows from this that we need both a measure of central tendency and one of variability or dispersion. In statistical work we would say that the first of our illustrations is a homogeneous group. The individuals within it are very similar in reference to the trait measured. The other group is described as heterogeneous, the variability being great. Both of these terms are widely used in educational, sociological, and psychological work.

MEASURES OF VARIABILITY

The Range

This statistic has already been defined as the high score minus the low score plus one. We used it as the first step in setting up frequency distributions. We will spend little more time on it here, other than to state that of all of the measures of variability this one is the most unreliable. By this we mean that from sample to sample this one varies more than does any of the others. An illustration will show why this is so. Suppose that we have a distribution of scores, the lowest of which is 30 and the highest is 103. The next score below 103 happens to be 90. By the use of our formula the range is found to be 74; but 13 of the points making up this range are the result of the large score of 103. The chances are good that the next sample will not contain this high deviate score and hence the range would be much smaller.

The range can be justifiably used when we want a hasty measure of variability and we do not have the time to compute one of the others. Each of the statistics discussed later in this chapter is a better measure. Of course, in dealing with a population instead of a sample, the range becomes more useful.

The Quartile Deviation

The quartile deviation, symbol Q, is frequently called the semi-interquartile range. In the previous chapter it was stated that there were two quartile points, Q_1, the equivalent of the 25th centile, and Q_3, the equivalent of the 75th centile. The quartile deviation or the semi-interquartile range is half of the distance between these two quartile points. In symbols we would write this as follows:

$$Q = \frac{Q_3 - Q_1}{2} \tag{5.1}$$

This statistic is easy to calculate. To illustrate we have again set up in Table 5.1 the distribution of geography scores previously used for illustration.

TABLE 5.1.
Computation of the Quartile Deviation

	f	
80–84	1	
75–79	1	
70–74	1	
65–69	4	7
60–64	4	
55–59	7	
50–54	6	
45–49	6	
40–44	6	10
35–39	3	
30–34	0	
25–29	1	

$N = 40$

We must first calculate Q_1 and Q_3. By definition, Q_1 is the point in this distribution which has 25 percent of the scores below it. Twenty-five percent of 40 cases is 10 cases.. We start counting cases from the bottom and we find that below the lower limit of the interval 45–49 we have exactly 10 cases. No interpolation is necessary in this case and we record that $Q_1 = 44.5$. To find Q_3 we repeat this same process except that instead of going up 75 percent of the cases, we come down from the top 25 percent of the cases.

Counting down we find that when we get to 64.5 (the lower limit of the interval 65–69), we have included 7 cases. We need 3 more. Since there are 4 cases in this interval we need to go three-fourths of the way through the interval. We can show the operation as follows:

$$\begin{aligned} Q_3 &= 64.5 - \frac{3}{4}(5) \\ &= 64.5 - \frac{15}{4} \\ &= 64.5 - 3.75 \\ &= 60.75 = 60.8 \end{aligned}$$

To find Q we proceed as follows:

$$\begin{aligned} Q &= \frac{Q_3 - Q_1}{2} \\ &= \frac{60.75 - 44.5}{2} \\ &= \frac{16.25}{2} \\ &= 8.125 = 8.12 \end{aligned}$$

Interpretation of Q. In the previous chapter we found that the median for the data in Table 5.1 was 52.8. In a normal distribution, if we take the median and add and subtract one quartile deviation on each side of it, we will cut off approximately 50 percent of the cases. That is, for the data at hand, we would expect to find 50 percent of the cases to fall between 60.9 (the median plus one Q) and 44.7 (the median minus one Q). This is shown graphically in Figure 5.1.

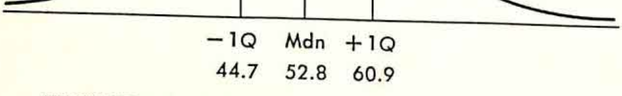


FIGURE 5.1 **Relationship of Q to the normal curve.**

Using the fact expressed in the above paragraph we can make a rough check on our calculations. According to our statement we would expect 50 percent of the cases in the distribution shown in Table 5.1 to fall between 44.7 and 60.9. Let us look at Table 5.1 and see how this checks out. Let us first find 44.7. It is almost at the bottom of the interval 45–49. The value 60.9 is roughly one-fourth of the way through the interval 60–64. There are 4 cases in this interval and we shall take 1 of them into our calculations.

How many cases are there between 44.7 and 60.9? We find 6 in the interval 45–49, 6 in the interval 50–54, 7 in the interval 55–59, and we decided that we could use 1 case in the interval 60–64. The sum of the frequencies between our points is 20 which is 50 percent of 40, the total number of cases.

It also happens that if we measure off 4 quartile deviations on each side of the median, we will include practically all of the cases. We can state this briefly by saying that 8 Q's approximately cover the range.

When to Use the Quartile Deviation. Since the quartile deviation is associated with the median, it follows that whenever the median is used as a measure of central tendency, the quartile deviation is the appropriate measure of variability. It may be recalled that the median is the statistic to be used as a measure of central tendency when we have a skewed distribution. Even when distributions are skewed, the check using the middle 50 percent of the cases will work.

The Average Deviation

This statistic, also referred to as the mean deviation, is no longer widely used in statistical work, having been replaced by the standard deviation. A brief consideration of it here, however, may make the material on the standard deviation, which follows, easier. To begin, we shall define the symbol x as a deviation of any score from the mean of its distribution. In symbols, we can write this as follows:

$$x = X - \bar{X} \tag{5.2}$$

where

x = the deviation of a score from the mean.
X = a raw score.
$\bar{X}$ = the mean.

In any distribution the sum of these deviations from the mean is equal to 0. This is an important fact about the mean. About no other point in any distribution is this true. A better definition of the mean would be that it is that point about which the sum of the deviations is equal to 0.[1]

Since the sum of the deviations about the mean is equal to 0, it follows that we can obtain no average deviation unless we modify our data. In actual practice, we sum the deviations disregarding the signs. The equation for the average deviation is written

$$AD = \frac{\Sigma|x|}{N} \tag{5.3}$$

where

$|x|$ = the absolute deviations of the scores from the mean.
N = the number of cases.

---

[1] The science student may note the resemblance of the mean to the fulcrum of a system which is in equilibrium. The product of the force or weight of each unit times its distance from the fulcrum is known as a moment, and the sum of the moments on one side of the fulcrum is equal to the sum of the moments on the other side of the fulcrum.

The calculation of the average deviation is shown in Table 5.2.

TABLE 5.2. Computation of the Average Deviation

X	x	
26	10	
24	8	
22	6	
20	4	
18	2	
16	0	
14	-2	
10	-6	
6	-10	
4	-12	
$\Sigma X = 160$	$\Sigma x = 0$	
$\bar{X} = 16$	$\Sigma\|x\| = 60$	$N = 10$

$$AD = \frac{\Sigma|x|}{N} = \frac{60}{10} = 6$$

The Standard Deviation

Of all the measures of variability, the standard deviation is by far the one most widely encountered, mainly because it is used in so many other statistical operations. We shall begin by showing how it is computed for both ungrouped and grouped data. With the ungrouped data, the process starts out in the same fashion as does that for the average deviation. That is, we first compute the mean. Then we compute the deviation of each score from this mean (see Table 5.3). Then we square each of these deviations and add this

TABLE 5.3. Computation of the Standard Deviation for Ungrouped Data

X	x	x^2
20	7.6	57.76
18	5.6	31.36
16	3.6	12.96
14	1.6	2.56
13	$.6 \quad \Sigma = +19$	.36
11	-1.4	1.96
10	-2.4	5.76
9	-3.4	11.56
8	-4.4	19.36
5	$-7.4 \quad \Sigma = -19$	54.76
$\Sigma = 124$	$\Sigma = 0.0$	$\Sigma = 198.40$
$\bar{X} = 12.4$		

column. A check on our work would be that the sum of the x column or the sum of the deviations about the mean should be 0. We find that this is true for our problem. To find the standard deviation we use the following formula and substitute in it as shown.[2]

$$s = \sqrt{\frac{\Sigma x^2}{N - 1}} \tag{5.4}$$

$$s = \sqrt{\frac{198.40}{9}}$$

$$s = \sqrt{22.04}$$

$$s = 4.7$$

If a calculating machine is available, it is much easier to use the so-called raw score formula for getting the sum of the deviations squared. This sum is usually referred to as the sum of the squares and is one of our most useful statistics. This technique is illustrated in Table 5.4. In this table we copy down the scores and label the first column X. The second column is merely the square of each X. (The student should remember that there is a table of squares in the back of the book from which the square of any number from 1 to 1000 is easily read.) Both columns are then summed. It should be pointed out that when a machine is used, it is not necessary to copy down the original scores or their squares as shown in Table 5.4. These are entered into the machine one at a time, and as the process goes along, the sum of each is accumulated. At the end, both values may be read directly from the machine as may the number of cases.

TABLE 5.4. Computation of the Standard Deviation Directly from Raw Scores

X	X^2
20	400
18	324
16	256
14	196
13	169
11	121
10	100
9	81
8	64
5	25
$\Sigma = 124$	$\Sigma = 1736$

---

[2] In some texts, the standard deviation is found by using N instead of $N - 1$. As will be discussed in Chapter 9, $N - 1$ is used to correct for a bias in the sample standard deviation. When the sample is small, it is better to use $N - 1$. In large samples, it makes little difference which is used.

For this method we obtain the sum of the squares by the following equation:

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{N} \tag{5.5}$$

$$\Sigma x^2 = 1736 - \frac{(124)^2}{10}$$

$$\Sigma x^2 = 1736 - \frac{15376}{10}$$

$$\Sigma x^2 = 1736 - 1537.6$$

$$\Sigma x^2 = 198.4$$

This value for the sum of the squares is the same as obtained by taking the deviation of each score from the mean. Even without a calculating machine students may find that this method is easier than the first. After we obtain the sum of the squares, we next insert our values into equation 5.4 and solve for the standard deviation.

Standard Deviation for Grouped Data. This technique is shown in Table 5.5. The beginning steps are the same as those we went through in calculating the mean for grouped data. Inspection of Table 5.5. shows that

TABLE 5.5. Computation of Standard Deviation for Forty Scores on a Statistics Test

(1)	(2) f	(3) x'	(4) fx'		(5) fx'^2	(6) $f(x' + 1)^2$
80–84	1	6	6		36	49
75–79	1	5	5		25	36
70–74	1	4	4		16	25
65–69	4	3	12		36	64
60–64	4	2	8		16	36
55–59	7	1	7		7	28
				+42		
50–54	6	0	0		0	6
45–49	6	−1	−6		6	0
40–44	6	−2	−12		24	6
35–39	3	−3	−9		27	12
30–34	0	−4	0		0	0
25–29	1	−5	−5		25	16
				−32		
	$N = 40$		$\Sigma = 10$		$\Sigma = 218$	$\Sigma = 278$

the process is identical up to and including the setting up of column (4). We start out by selecting an arbitrary reference point. In this problem we chose the midpoint of the interval 50–54. Then we gave this interval a deviation value of 0 and set up our x' column (column (3)) as shown. We obtain the value in column (4), fx', by multiplying the values in the two previous columns. We obtain the fx'^2 values in column (5) by multiplying the values in the two previous columns. Then we sum the various columns as shown.

To find the standard deviation we again have to find the sum of the squares. For grouped data this is done as follows:

$$\Sigma x^2 = i^2 \left[\Sigma f x'^2 - \frac{(\Sigma f x')^2}{N}\right] \tag{5.6}$$

$$\Sigma x^2 = 5^2 \left[218 - \frac{(10)^2}{40}\right]$$

$$\Sigma x^2 = 25 \left[218 - \frac{100}{40}\right]$$

$$\Sigma x^2 = 25\,[218 - 2.5]$$

$$\Sigma x^2 = 25\,[215.5]$$

$$\Sigma x^2 = 5387.5$$

We obtain the standard deviation in the usual fashion by solving as shown below:

$$s = \sqrt{\frac{\Sigma x^2}{N-1}}$$

$$s = \sqrt{\frac{5387.5}{39}}$$

$$s = \sqrt{138.13}$$

$$s = 11.8$$

We can check the accuracy of the work in several ways. The first of these, called Charlier's check, is a check upon the accuracy of the sums in our fx' and fx'^2 columns. To carry out this check, we set up column (6) as shown in Table 5.5. The heading here tells us to take each x' value, add one to it, square it, and then multiply it by the frequency of the interval. For the top interval this becomes 6 plus 1 which is 7, 7 squared is 49, and this multiplied by the frequency, 1, is still 49. When each interval has been so treated, the values in column (6) are summed. Charlier's check is made by substituting in the following equation:

$$\Sigma f(x+1)^2 = \Sigma f x'^2 + 2\Sigma f x' + \Sigma f \tag{5.7}$$

$$278 = 218 + 2(10) + 40$$

$$278 = 218 + 20 + 40$$

$$278 = 278$$

We shall present a second, rough check on our computed standard deviation after we discuss the meaning of a standard deviation.

Interpretation of Standard Deviation. This statistic is associated with the normal curve. When our data take the shape of the normal curve, standard deviation units measured off along the base line, starting from the mean, always cut off certain proportions of the area under the curve. In our problem that is illustrated in Figure 5.2 the mean of 53.2 is recorded at the center of the curve. Then one standard deviation unit is added to this mean, 53.2 + 11.8, resulting in a score of 65.0. Similarly, one standard deviation is measured off on the other side of the mean. In a normal curve, these two standard deviation units taken together cut off approximately 68 percent of the area. We shall in future chapters refer to this as two-thirds the area. If we measure off two standard deviation units on each side of the mean, we include between these two points approximately 95 percent of the area. And when we take three standard deviation units, over 99 percent of the area of the curve is included. Actually about 13 cases in 10,000 are left over on each side of the mean out beyond the plus and minus three standard deviation points. In a later chapter on the normal curve we shall see where these values come from.

Let us return to Table 5.5. and see how our computed standard deviation checks when we count the number of cases included by one standard deviation unit on each side of the mean. As previously noted, the points corresponding to plus and minus one standard deviation are 65.0 and 41.4, respectively. Let us locate 65.0 on Table 5.5. For all practical purposes we might say that it is the point at the top of the interval 60–64. So we shall start counting down the number of cases between this and 41.4. We have to determine how many of the 6 cases in the interval 40–44 are above 41.4 and are to be included in our sum. The top of this interval is 44.5. The value 41.4 is 3.1. units below this. Hence we want 3.1/5 of the 6 units or approximately 4 of them. So starting from the top (65.0) we add 4 + 7 + 6 + 6 + 4 which equals 27. The number of the cases in this problem is 40. A standard deviation taken on each side of the mean would then include $^{27}/_{40}$ of the distribution which in terms of percents reduces to between 67 and 68 percent. This check may not always come as close as this to the values of the normal curve; but if the computational work is correct, the percentage areas of the normal curve should be approximated.

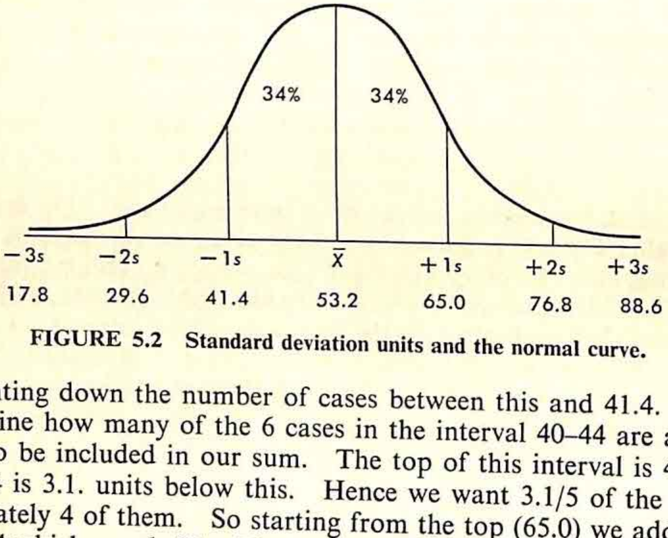


FIGURE 5.2 Standard deviation units and the normal curve.

Relation of the Range to the Standard Deviation. We noted above that approximately six standard deviations cover the range. This is true only when the number of cases is large. As N decreases, the number of standard deviations needed to include all of the cases decreases. Here is how it looks:[3]

---

[3] Adapted from L. H. C. Tippett. On the extreme individuals and the range of sample from a normal population. *Biometrika*, 1925, **17**, 386.

N	*Number of Standard Deviations Including the Range*
5	2.3
10	3.1
25	3.9
30	4.1
50	4.5
100	5.0
500	6.1
1000	6.5

The Variance. Much of our statistical work is handled using another statistic to describe variability. This statistic is called the variance and is nothing more than the standard deviation squared.

$$s^2 = \frac{\Sigma x^2}{N - 1} \tag{5.8}$$

where s^2 = the variance and other symbols are as previously described. In this text we shall have limited use for the variance, but a good share of modern statistics is based upon the manipulation of variances.

The Error of Grouping. In Chapter 4 we noted that when data are grouped the means computed from such data are not identical with the means computed from the raw scores. We stated that these discrepancies were caused by the error of grouping. We shall now examine this phenomenon more closely.

Suppose that we have a distribution with a range from 30 to 100 and that these scores are set up into a frequency distribution as shown in Figure 5.3.

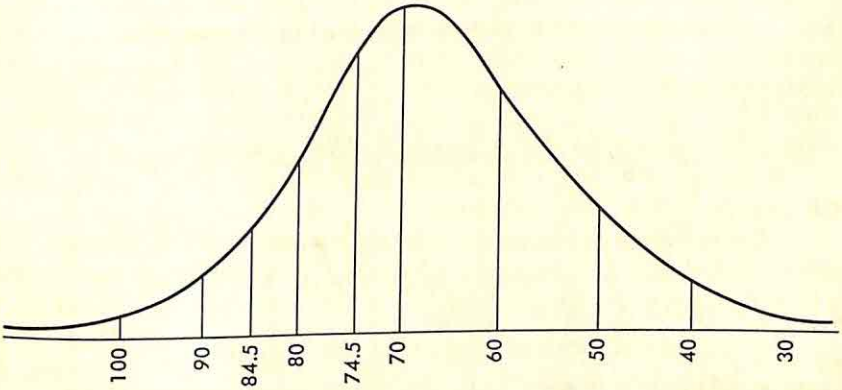


FIGURE 5.3 Frequency distribution and curve for data showing that the midpoint of the interval does not have an equal area on both sides of it.

Note that all of the data are included within seven class intervals. Let us suppose that the mean is 70. First, we shall consider the interval 70–79. The midpoint of this interval is 74.5. In Figure 5.3 it is apparent that more of the area of this interval is found below the midpoint than above it. By chance, we should expect to find more scores between 69.5 and 74.5 than between 74.5 and 79.5. This would be true for all intervals above the mean. The midpoint of each interval is too large when taken as an average of all of the scores in an interval. An inspection of the intervals below the mean shows that for all of these intervals, the reverse of this is true. The midpoint for these intervals is too small or too low.

In the computation of the mean, these discrepancies are of no major importance. Deviations on the positive side tend to be canceled out by deviations on the negative side and, in the long run, the mean computed by the group method is very similar to that computed by the raw score method. This, however, is not true for the standard deviation. The coarser the grouping, the greater the effect of this error of grouping upon the standard deviation. Any distribution which has less than 12 to 14 class intervals can be considered to have coarse groupings, and the standard deviation computed from such frequency distributions should be corrected. The pertinent correction here is known as Sheppard's correction and is given by the following formula:

$$s_c = \sqrt{s^2 - \frac{i^2}{12}} \tag{5.9}$$

where

s_c = the standard deviation corrected for the systematic error brought about by coarse grouping.
s^2 = standard deviation squared, or variance, obtained from the coarsely grouped data.
i = the size of the interval.

Averaging Standard Deviations. In Chapter 4 we learned that if we wished to average two or more means, we could do it by the so-called weighting method. Standard deviations cannot be treated in the same manner. When two or more standard deviations are to be averaged, the following formula from McNemar (**10**) is to be used:

$$s_T = \sqrt{\frac{N_A(\bar{X}_A{}^2 + s_A{}^2) + N_B(\bar{X}_B{}^2 + s_B{}^2)}{N_A + N_B} - \bar{X}_T{}^2} \tag{5.10}$$

where

s_T = standard deviation of combined group.
N_A, N_B = number of individuals in each of the two groups.
$\bar{X}_A$, $\bar{X}_B$ = means of the two groups.
$\bar{X}_T$ = weighted mean of the two groups combined.
s_A, s_B = standard deviations of the two groups being combined.

If more than two groups are being combined, an additional element is placed in both the numerator and denominator for each additional group.

Summary of the Measures of Variability. At this point we shall briefly summarize the four measures of variability.

1. *The Range.* This is least reliable of all four measures. It is of limited use except when speed is an issue or in simple situations such as setting up a frequency distribution.

2. *The Quartile Deviation.* This statistic is always associated with the median, being used whenever the median is used as a measure of central tendency. This is usually the case in skewed distributions. It is noted here that in a normal distribution the quartile deviation and the standard deviation have a constant relationship. $Q = .6745s$.
3. *The Average Deviation.* This is used with the mean in a normal distribution. In the past it was a rather widely used statistic, but today it is almost completely replaced by the standard deviation.
4. *The Standard Deviation.* This is the most reliable of all four measures and the one most frequently encountered. It is associated with the mean and, when used, we assume that the shape of our distribution of scores or measurements tends to conform to that of the normal curve. As we shall see this statistic has many uses in modern statistics and is one of our most important ones.

Saving Time in Computational Work. Many times we are plagued with decimals, negative numbers, and rather large numbers. Our computational work can be made much easier if we become aware that there are operations which can be performed on scores which do not affect the results and which make our work much easier. Suppose we have a group of scores, some of which are negative. Let us add a constant to each of the scores so that all scores are positive. What have we done to the mean and the standard deviation of our original scores? In another case, suppose that our scores are all decimals to the nearest thousandths. Suppose that we multiply each score by 1000 (this is the same as dropping the decimal point). How are the mean and standard deviation affected this time? Such operations as these are known as coding and are illustrated in Table 5.6.

TABLE 5.6. Coding Data in Various Ways

	(1) X	(2) x	(3) x^2	(4) $(X - 10)$	(5) x	(6) x^2	(7) $2X$	(8) x	(9) x^2
	20	5	25	10	5	25	40	10	100
	19	4	16	9	4	16	38	8	64
	18	3	9	8	3	9	36	6	36
	17	2	4	7	2	4	34	4	16
	16	1	1	6	1	1	32	2	4
	15	0	0	5	0	0	30	0	0
	14	−1	1	4	−1	1	28	−2	4
	13	−2	4	3	−2	4	26	−4	16
	12	−3	9	2	−3	9	24	−6	36
	11	−4	16	1	−4	16	22	−8	64
	10	−5	25	0	−5	25	20	−10	100
Sums =	165	0	110	55	0	110	330	0	440

We shall now take up Table 5.6 step by step. In column (1) is a set of 11 scores, the sum of which equals 165. Dividing this by the number of cases, 11, gives us a mean of 15. In column (2) we have the deviation of each of the scores from the mean of 15. This column sums to 0, which is correct, as the sum of the deviations about the mean is equal to 0. In column (3) we have squared each of our deviations and these are summed, giving us a sum of the squares equal to 110. Substituting into the formula for standard deviation and solving results in a standard deviation of 3.3.

In column (4) appear the results of subtracting a constant of 10 from each of the scores in column (1). These are added and the mean is again computed. This time the mean is 5. Columns (5) and (6) are the deviations and the deviations squared for each of the values in column (4). This time the sum of the squares is again 110 and the standard deviation is also found to be 3.3. Then we see that when we subtracted a constant from each of our scores, the same constant was subtracted from the mean, but the standard deviation remained the same. So to get the correct answer when data are coded in this fashion, we would have to add to the mean the constant which was subtracted from each score, but the standard deviation does not have to be changed. If we added a constant to each score in any distribution, we would find that the results are the same as above. The mean has changed, but the standard deviation remains the same. In this case, however, instead of adding the constant to the mean, we would have to subtract it to obtain the correct mean.

Now let us see what happens when we multiply each score in a distribution by a constant. In column (7) appear the same scores as in column (1) except that each has been multiplied by a constant of 2. This column is summed, the sum divided by 11, and a mean of 30 obtained. In columns (8) and (9) are found the deviations and the deviations squared of each of the scores in column (7) from this mean of 30. Computing the standard deviation gives 6.6. Hence, when every score in a distribution is multiplied by the same value, both the mean and standard deviation were multiplied by the same value. This time if we divide our mean of 30 and our standard deviation of 6.6 by 2 we have the mean and standard of the original data. If instead of multiplying each score by a constant we divide, we shall find that both the mean and the standard deviation are divided by the same constant. In this case to correct for our coding, the mean and standard deviation would both have to be multiplied by the constant we used in dividing the scores. It is to the student's advantage to remember these coding effects and to use them as often as possible.

Computing statistics by grouping data uses these coding operations. We shall illustrate this using the data in Table 5.7. In this Table column (1) includes the frequencies for each of the intervals. In column (2) are written the midpoints of each of the intervals. In calculating the mean by the method of the arbitrary reference point, the first step is to choose the midpoint of one of the intervals as the starting point. In this case we have selected the interval 50–54, the midpoint of which is 52. This midpoint of 52 is then subtracted from each of the midpoints recorded in column (2). The remainders are the values shown in column (3). We are continuing the coding by dividing each of the values in column (3) by 5, the size of the class interval. These quotients make up column (4). The numbers in column (5) are the

TABLE 5.7. The Use of Coding When Data Are Grouped

	(1) f	(2) Midpoints	(3) Midpoints Minus 52	(4) Values in Column Divided by $5(x')$	(5) fx'
80–84	1	82	30	6	6
75–79	1	77	25	5	5
70–74	1	72	20	4	4
65–69	4	67	15	3	12
60–64	4	62	10	2	8
55–59	7	57	5	1	7 $\Sigma = +42$
50–54	6	52	0	0	0
45–49	6	47	−5	−1	−6
40–44	6	42	−10	−2	−12
35–39	3	37	−15	−3	−9
30–34	0	32	−20	−4	0
25–29	1	27	−25	−5	−5 $\Sigma = -32$
	$N = 40$				$\Sigma = 10$

products of the values of column (1) multiplied by the values in column (4). Then these are summed, giving us in this case a sum of 10.

Now let us find the mean of this coded sum: 10 divided by 40 gives us a mean value of .25. Now we decode this value by reversing the steps that we went through in the coding process. We first multiply this by 5 (.25 × 5 = 1.25) and then we add this to 52, the value we subtracted out. If we add 1.25 to 52 we get 53.2, the value that was obtained in Chapter 4 for the mean of these data. Instead of going through the operations in this manner, we use equation 4.2 which is as follows: $\bar{X} = M' + \frac{\Sigma fx'}{N}(i)$. All of the decoding operations are summarized in this formula.

Exercises

1. The instructor of a freshman English course in a university has administered the *STEP Writing Test* to one of his classes. Their scores are as follows:

37
35
43
45
45
35
35
42
40
46

35
34
44
43
34
40
38
37
40
38
36

(a) What is the range of these scores?
(b) Why might the instructor want to know the range?
(c) The instructor reads this chapter and says "For my purposes the range is the best measure of variability—unreliability be hanged!" Is he right or wrong? Why?

2. The senior class guidance counselor of a large high school decides to give the *Cooperative School and College Ability Test* (*SCAT*) to 112 seniors who are planning to go to college. The students take the tests during study periods. The first group achieves these scores:

30
75
55
73
83
62
61
73
24
54
58

(a) What is the range of the distribution?
(b) The principal asks the counselor to estimate the range of the total group of seniors who plan to enter college. Is the range of this first group the best estimate?

3. The distribution of scores for the 112 seniors mentioned in Exercise 2 is:

90–94	1
85–89	2
80–84	3
75–79	5
70–74	11
65–69	12
60–64	10
55–59	14
50–54	11
45–49	11
40–44	16
35–39	8
30–34	5
25–29	2
20–24	1

(a) What is the quartile deviation of this distribution?
(b) How might this Q help the counselor in his work?

4. On a scale to measure attitudes toward school integration a senior civics class made the following scores:

6
4
8
1
3
4
9
6
5
2
8
5
1
7
3
8
3
7
1
5

(a) What is the quartile deviation?
(b) Is quartile deviation appropriate here?

5.
(a) What is the standard deviation of the scores given in Exercise 3?
(b) How would you expect the standard deviation of *SCAT* scores on a national sample of 1500 students to compare with this?
(c) Use Charlier's check to be sure your computations are correct.

6.
(a) Use the raw score formula to compute the standard deviation for the scores given in Exercise 1.
(*b*) Multiply this s by 2 and add the product to the mean of the distribution. What proportion of the scores in another similar sample would you expect to be lower than this figure?

7. The same statistics test was administered to three classes with the following results:

	$\bar{X}$	s	N
Class A	82	10	25
Class B	86	12	35
Class C	88	16	20

   (a) What is the mean of the three groups combined?
   (b) Find the standard deviation of three groups taken together.

8. A group of university seniors made the following scores on the *Graduate Record Examination*:

602
650
552
402
460
501
550
484
462
544
542
567
560
481
498
422
580
601
590
420
449
486
512
498
407
558
550
594
575
476
470
520

   By the most efficient method compute the mean and standard deviation.

9. Calculate the standard deviation of any or all of the problems at the end of Chapter 4.

6

CENTILES AND STANDARD SCORES

CENTILES

In this chapter we shall consider two types of scores very widely used in educational and psychological work. The first of these, the centile, or percentile point, we have already met in Chapter 4. There it was mentioned that the median was the 50th centile and, since we were learning to compute the median, we studied the computation of any centile point. To establish a set of centile norms, that is, to compute each of the centile points for any set of data, would be a rather tedious job. Fortunately we can construct a special type of curve from which these centile points may be read.

Obtaining the Cumulative Frequencies. In Table 6.1 appears a frequency distribution of the scores of 376 boys on a test of mechanical ability. Column (2) contains the frequencies. In column (3) we have the cumulative frequencies, *cf.* These are obtained in this manner. Starting from the bottom, we ask ourselves how many scores there are below the upper limit of the bottom interval. The answer is 4. So a 4 appears as the bottom entry in column (3). Next we ask how many scores fall below the upper limit of the next interval, that is, how many scores are there below 14.5. The answer to this is 19, the sum of all of the scores below this point. The cumulative frequency for the next interval is 23 + 19 or 42. The process is continued by adding the frequency of the interval to the cumulative frequency of all the intervals below the given interval. If the work is done correctly, the *cf* of the top interval will be the same as the number of cases.

Obtaining the Cumulative Proportions and Percentages. As pointed out in Chapter 3, the easiest way to convert a set of frequencies to proportions is to obtain a constant multiplier by dividing 1 by the number of cases and then multiplying each frequency by this constant to obtain the proportion. For these data, the constant multiplier is 1 divided by 376 which equals .00266. Each cumulative frequency in column (3) is then multiplied by .00266 and the

TABLE 6.1. Obtaining Cumulative Frequencies, Proportions, and Percentages

(1)	(2) f	(3) cf	(4) cp	(5) cP
60–64	2	376	1.000	100
55–59	12	374	.995	99.5
50–54	20	362	.963	96.3
45–49	32	342	.907	90.7
40–44	46	310	.824	82.4
35–39	58	264	.702	70.2
30–34	64	206	.548	54.8
25–29	58	142	.377	37.7
20–24	42	84	.223	22.3
15–19	23	42	.112	11.2
10–14	15	19	.050	5.0
5–9	4	4	.011	1.1
	$N = 376$			

results are recorded in column (4). This column is headed *cp*, cumulative proportions. To change these cumulative proportions to cumulative percentages, *cP*, each *cp* is multiplied by 100. These values appear in column (5).

Constructing the Cumulative Percentage or Ogive Curve. In setting up the cumulative proportion, cumulative percentage, or ogive curve as it may be called, we follow the general rules of building graphs which we discussed in Chapter 3. We attempt to have the ratio of the Y-axis to the X-axis as 2 is to 3. The cumulative proportions or cumulative percentages are always placed along the vertical or Y-axis. As in the frequency polygon, the scores are entered on the horizontal or X-axis. The values on the Y-axis will always range between 0 and 1 or 0 and 100, depending on whether or not we are using proportions or percentages. In this case we are going to plot the cumulative percentages. It may be recalled that when we were plotting a frequency polygon we placed the tally mark above the midpoint of the class interval. *In plotting the cumulative percentage graph, we use the upper limits of each of the intervals and place our tally marks above these.* This is consistent with the concept of cumulative frequencies or cumulative percentages. In setting these up in our frequency table, we asked ourselves how many or what percent of the cases fell below the upper limit of the class interval. The setting up of one of these curves is shown in Figure 6.1.

The upper limit of the bottom interval is 9.5. Hence, above 9.5 on the X-axis we go up 1.1 points and place our tally. Above 14.5, the upper limit of the next interval, we count off 5 units on the Y-axis and again place our mark. This process is continued until all values are plotted. Then a smoothed curve is drawn. Some of the plotted points will not be on the curve, some appearing on one side and some on the other. Such smoothed curves as this always take the shape as shown in Figure 6.1 and are frequently referred to as S-shaped curves. The curve is brought to the base line by going down to the next interval and giving it a *cP* of 0.

Reading the Centile Points from the Ogive Curve. If the cumulative percentage curve has been accurately constructed on a large piece of graph paper, centile points may be read from it with a high degree of accuracy. To read the points a ruler is placed on the *cP* column at the centile point desired and a straight line is drawn over to the curve. From this point on the curve another line is drawn down to make a right angle with the *X*-axis. The point on the *X*-axis where this vertical line meets the *X*-axis is the desired centile point. In Figure 6.1 lines are drawn showing the values for C_{80}, the median, and Q_1. These values are approximately 43, 33, and 25, respectively. These values are very close to the computed values for the same statistics which are 43.5, 33.4, and 25.4.

Use of Centiles. Centiles are widely used in educational circles in reporting the results of standardized tests. In their favor it can be said that they are very easy to understand. Even if one does not know that a person who has a centile score of 77 is at a point above 77 percent of those who took a certain test or upon whom the test was standardized, at least the 77th centile looks like 77 percent and even the most uninformed teacher can understand what that means. Since these appear to be like percents, there is little difficulty in understanding their meaning. Also they very adequately give an indication of an individual's rank in a group.

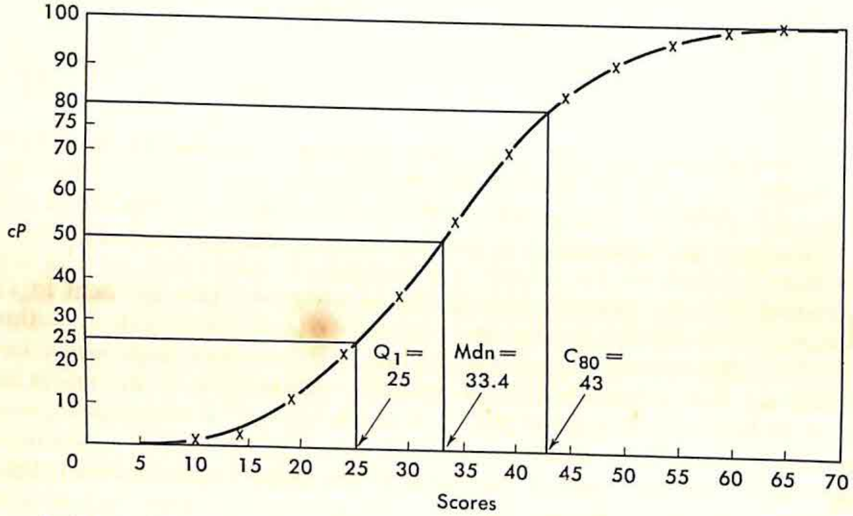


FIGURE 6.1 **Cumulative percentage or ogive curve for the ability test scores in Table 6.1.**

Centiles do, however, have serious limitations and many testmakers and users of tests no longer bother with them. If you examine a set of centile norms or a profile sheet based upon such norms, you will note that the centile norms are piled up at the middle of the distribution. Notice Figure 6.2. A raw score of 33 is equivalent to C_{50}, a raw score of 36, to a centile point of 60, and a raw score of 30, to a centile point of 40. A change in six raw score units is the equivalent to a change in 20 centile units. There is, then, a piling up of centile points at the center of the distribution and differences between them at this part of the curve have little meaning.

To state that an individual who is at C_{47} on a certain test differs from the individual who is at C_{51} on the same test is making much ado about nothing. At the center of distribution, the use of centile scores tends to exaggerate differences that are not actually in existence. Centiles are unequal units of measurement and cannot be treated arithmetically. That is, there is no justification for averaging them, combining them, or treating them in any mathematical fashion. As statistics go, these are dead ends. Nothing should be done with them. If one wishes to manipulate data which have been reduced to centiles, one should convert the data back to raw scores and work with these. Since centiles are unequal units of measurement, some statistical workers feel that we would be better off without them. As far as standardized tests go, they are becoming less and less frequent as a method of reporting norms for such tests. However, in certain circles, they will be with us for a long time to come.

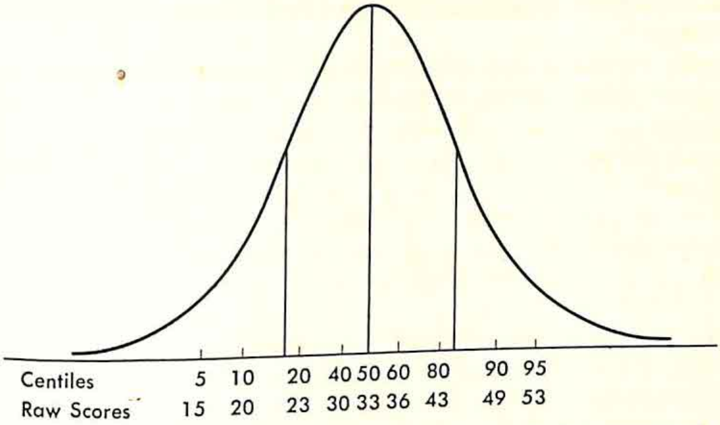


FIGURE 6.2 Centile and raw score equivalents for the data in Table 6.1.

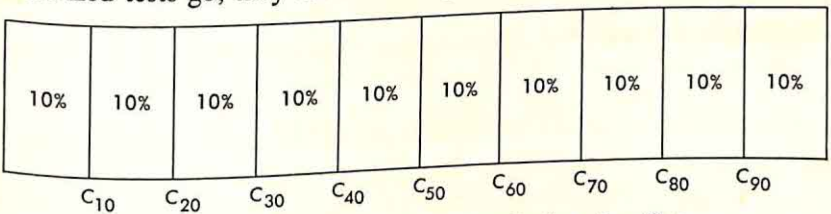


FIGURE 6.3 The rectangular distribution of centiles.

Let us see why centiles pile up at the center of the distribution. If we go back to the original definition of a centile point as a point in a distribution with a certain percent of the distribution of the cases below it, it follows that C_{10} is that point in the distribution with 10 percent of the cases below it. Ten percent of the area of a curve falls below C_{10} would be another way of stating this. Figure 6.3 illustrates this. If we take C_{20} next, it is by definition that point with 20 percent of the cases in the distribution below it. Ten percent more of the area is added below C_{20} in our figure. This may be continued until all of the area or all of the cases of a distribution are accounted for. The resulting figure looks like Figure 6.3 which instead of being a normal curve is a rectangle. The distribution of centiles is described as being rectangular. Measurements made in social and biological sciences tend to take the form of the normal curve. Distortions come about when such data are manipulated using centiles which belong with a distribution of another shape.

Frequently we would like to know the relationships that exist between the various centile points and the normal curve. These are shown in Figure 6.4.

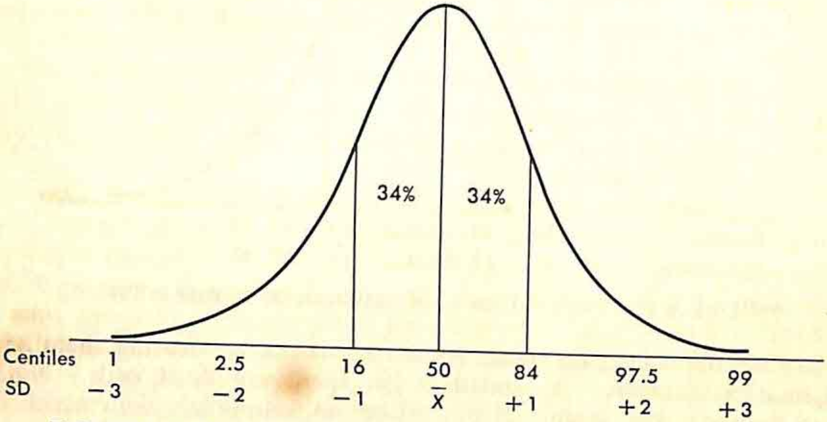


FIGURE 6.4 Centile equivalents of the various standard deviation units.

One standard deviation above the mean is the equivalent of C_{84}. This means that 84 percent of the cases in a normal distribution are below the point equivalent to one standard deviation above the mean. Other relationships can be seen from Figure 6.4.

STANDARD SCORES

The second type of score to be considered in this chapter is the standard score, the symbol of which is z. The formula for a standard score is given below:

$$z = \frac{X - \bar{X}}{s} = \frac{x}{s} \tag{6.1}$$

where

X = any raw score.
$\bar{X}, s$ = the mean and standard deviation of the distribution of scores, respectively.

To illustrate the computation and nature of standard scores let us take the following scores which are a part of a distribution with a mean of 60 and a standard deviation of 10.

X	x	z
70	10	1.00
60	0	.00
50	−10	−1.00
54	−6	−.60
46	−14	−1.40

In the first column we have the raw scores (X). Each of these is subtracted from the mean and then this deviation from the mean, or x, is divided by the standard deviation to change the deviation values into standard score values. The raw score of 60 is at the mean. There is no deviation; hence the standard score is 0. A raw score of 70 is one standard deviation above the mean. This results in a z-score of 1.00. When we change raw scores to standard scores we are expressing them in standard deviation units. These standard scores tell us how many standard deviation units any given raw score is above or below the mean.

Since three standard deviations in either direction on either side of the mean include practically all of the cases, it follows that the highest z-score usually encountered is $+3.00$ and the lowest is -3.00. We can describe the distribution of such z-scores by saying that they have a mean of 0 and a standard deviation of 1. This is shown in Figure 6.5. Thus any time that we see a standard score we should be able to place exactly where an individual falls in a distribution. A student with a z-score of 2.50 is 2.5 standard deviations above the mean in that test distribution and has a very good score. These standard scores are equal units of measurement and hence can be manipulated mathematically. It should be noted here also that changing a distribution of scores to z-scores does not change the shape of the original distribution of scores. If the distribution was positively skewed to begin with, z-scores made from such a distribution would be positively skewed.

Since z-scores are expressed in decimals and since in any distribution half of them are negative, they are rather cumbersome to handle. Many times so-called linear transformations are made. Such transformations consist of making the scale larger so that negative scores are eliminated and using a larger standard deviation so that decimals are done away with. This transformed score can be obtained from the following equation:

$$\text{Standard score} = z\,(\text{new standard deviation}) + \text{the new mean}$$

A common form for these transformations is based upon a mean of 50 and a standard deviation of 10. In equation form this becomes

$$\text{Standard score} = z(10) + 50$$

or starting with the raw scores we have

$$\text{Standard score} = \frac{(X - \bar{X})}{s}(10) + 50$$

This system with a mean of 50 and a standard deviation of 10 is a very popular one. It has been widely used by branches of the Armed Forces for many years. With this distribution we have a mean of 50 and a range of between 20 and 80. All scores are positive and all can be rounded to two-place numbers. Figure 6.5 shows this distribution of scores and also several others. The third row in Figure 6.5 is the type used on the *Army General Classification Test* of World War II. This has a mean of 100 and a standard deviation of 20. The last row of these figures shows the type used by the College Entrance Examination Board and the *Graduate Record Examination.* Here we note a mean of 500 and a standard deviation of 100. Any system can be set up, but the ones noted above are those most frequently encountered.

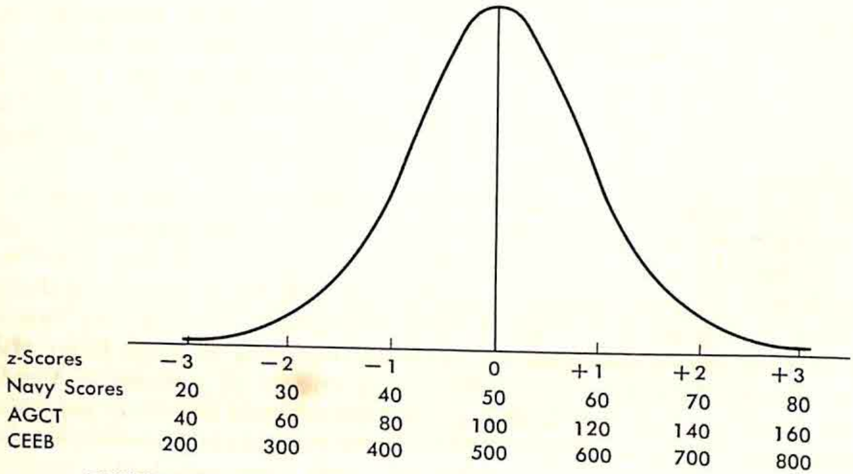


FIGURE 6.5 Distribution of the various types of standard scores.

Some standard scores have been normalized. By this we mean that the distributions of these scores has been made to conform to that of the normal curve. We shall take up the normalizing of scores in a later chapter. Here we shall briefly note some of these normalized standard scores. A very common one is the *T*-score which has a mean of 50 and a standard deviation of 10. Then there are stanines (standard nines) which have a mean of 5 and a standard deviation of 2. One familiar with any of Wechsler's tests of mental ability has encountered another type of standard score called a deviation *IQ*. These are standard scores with a mean of 100 and a standard deviation of 15.

Uses of Standard Scores. Since standard scores are equal units of measurement and since their size is the same from distribution to distribution, they become a very useful tool both in the reporting of test scores and in doing research using test results. When the results of different tests taken by the same individuals are to be compared, this is best done by the use of standard scores. This process is illustrated in Table 6.2.

In Part A of Table 6.2 are presented the scores of students on three elementary school tests. Only the scores of three students are shown. At the

TABLE 6.2. Comparing and Combining Scores Made on Different Tests by the Use of Standard Scores

Part A. Raw Scores

Student	Geography	Spelling	Arithmetic
A	60	140	40
B	72	100	36
C	46	110	24
etc.			
Mean	60	100	22
Standard Deviation	10	20	6

Part B. Standard Scores

Student	Geography	Spelling	Arithmetic	Average
A	50	70	80	67
B	62	50	73	62
C	36	55	53	48
etc.				

bottom are shown the mean and standard deviation of each test. If one looks at these data in Part A of Table 6.2, it should be apparent that as they stand, they convey little meaning. Which student had the best overall performance? On which test did the different students do best? or worst? None of these questions can be answered as the scores are shown.

Suppose that we now change these scores to standard scores. For this we shall use the system with a mean of 50 and a standard deviation of 10. We shall start with the geography test and for all of these we shall use this transformation equation

$$\text{Standard score} = \frac{X - \bar{X}}{s}(10) + 50$$

Taking the geography score for student A, his standard score for this test is

$$\begin{aligned} \text{SS} &= \frac{60 - 60}{10}(10) + 50 \\ &= 0 + 50 = 50 \end{aligned}$$

The geography score for student B becomes

$$\begin{aligned} \text{SS} &= \frac{72 - 60}{10}(10) + 50 \\ &= \frac{120}{10} + 50 \\ &= 12 + 50 = 62 \end{aligned}$$

This is continued until all scores are transformed. The results of transforming the scores in Part A of Table 6.2 are shown in Part B. This may seem to be a laborious and time-consuming process. However, if there are a lot of these to be transformed, it is most convenient to arrange the scores from high to low, find the deviation of each from the mean, and since these are in order, the rest of the process becomes very easy. Scores with a negative deviation are the same distance below the new mean as the positive ones with the same deviations above it. Actually one only has to compute transformed scores for deviations on one side of the mean.

Let us now examine Part B of Table 6.2. We now note that student A is at the mean in geography, two standard deviations above the mean in spelling, and three standard deviations above the mean in arithmetic. His average performance on these three tests was 67, one and seven-tenths standard deviations above the mean. For student A we can then say that his performance is average in geography, excellent in spelling, and superior in arithmetic. In this manner we can consider the achievement of each of the students on each of the three tests and we can get an average measure of his performance on all three tests. It should be noted that the only justifiable manner to compare scores is to first change the scores to standard scores and then do the comparing. Standard scores change the raw scores to equal and comparable units.

Another use of standard scores is in determining final grades for a course. Let us take a course which has three examinations of an hour each during the semester and a final examination of two hours' length. At the end of the semester the instructor averages the three hour-examinations and then combines this average with the final examination in some manner or other. This method is not correct. Suppose that the means and standard deviations of the three hourly examinations were as follows:

	First Test	*Second Test*	*Third Test*	*Final Exam.*
$\bar{X}$	52	87	62	124
s	8	17	11	21

If a student's scores on these three hourly examinations are averaged, each will not contribute equally to the average. The second test with the largest standard deviation would contribute more to the final average than the other two, and the first examination would contribute least. It is the size of the standard deviation which determines the contribution of each score to the final average.

In grading situations, if the teacher wants each of a series of tests to contribute equally to the final grade, it follows that the scores of individuals on each test should be changed to standard scores and that these should be averaged. Then each hourly test is contributing equally to the final grade. Suppose now that the instructor wants the final examination to have twice as much weight as each of the other tests in the final grade. These final grades should also be changed to standard scores. Then for any student his overall average is determined by taking his standard score on each of the first three tests, adding these, then adding to this sum two times his standard score on the final examination and dividing the total by five. In symbols:

$$\frac{z_1 + z_2 + z_3 + z_f(2)}{5}$$

where

z_1, z_2, z_3 = the standard scores on the three hourly examinations.
z_f = standard score on the final examination.
2 = weight final examination is to have.

By doing the above, each examination contributes the amount that the instructor desires it to. Some teachers convert all scores to standard scores before entering them into their class books. This is strongly recommended.

Science teachers and some others have special problems with laboratory grades, project grades, and the like. It is desired that the laboratory grades, for example, contribute a certain amount to the final grade. This can only be done with any accuracy by changing these to standard scores and giving them the desired weight in the final average as shown above for the combination of hourly examinations and finals.

Exercises

1. The frequency distribution of the scores of 127 third grade students on a vocabulary test is as follows:

Score	*f*
48–51	4
44–47	3
40–43	7
36–39	8
32–35	16
28–31	19
24–27	21
20–23	17
16–19	11
12–15	8
8–11	7
4–7	4
0–3	2

   (a) Construct an ogive curve for these data.
   (b) From the ogive read the median, the two quartiles, and the decile points.
   (c) Draw two frequency polygons, on the same axes, showing the distribution of raw scores and centiles.

2. What is the z-score equivalent of the following scores in the above distribution: 5, 18, 30, 39, 46, and 51?

3. Transform each of these to a standard score with a mean of 50 and a standard deviation of 10.

4. A psychologist has developed a battery of tests for selecting job applicants for a certain job. The tests and their means and standard deviations are:

	M	*SD*
Intelligence	100	12
Motivation	65	7
Manual Dexterity	56	5

The psychologist knows from his research that motivation is twice as important as intelligence for success on the job. He also knows that manual dexterity is twice as important as motivation for the job. Ten people apply for work. Their scores are as follows:

Applicant	*Intelligence*	*Motivation*	*Manual Dexterity*
A	113	65	61
B	98	42	52
C	101	91	58
D	82	50	55
E	143	73	63
F	107	60	54
G	100	62	42
H	99	66	54
I	113	59	56
J	100	68	56

Three vacancies exist. Which three applicants should be hired on the basis of these tests?

5. Does the difference between C_{50} and C_{55} (5) equal the difference between C_{90} and C_{95} (5) on an absolute scale of a measured trait?

6. An anthropologist finds that he is at the C_{98} point in height among one African tribe and at the C_{30} point among another tribe. Has he discovered anything about his height? Has he discovered anything about the relative height of the two tribes?

7

THE NORMAL CURVE

GENERAL NATURE OF THE NORMAL CURVE

For some time now we have been talking about the normal curve. In this chapter we shall examine it in detail, discuss its characteristics, its properties, and some of the basic ways in which it can be used. In the eighteenth century gamblers were interested in the chances of beating various gambling games and they asked mathematicians to help them out. DeMoivre (1733) was the first to develop the mathematical equation of the normal curve. Gauss and LaPlace (early nineteenth century) further developed the concept of the curve and probability. It was at about the same time that errors of observation made by astronomers were represented by a curve of this type. Today the normal curve is referred to as the curve of error, the bell-shaped curve, the Gaussian curve, or DeMoivre's curve.

By now the shape of this curve is familiar to all of you. Its maximum height is at the mean. In the language of the curve we say that the maximum ordinate (ordinates are given the symbol y) is at the mean. All other ordinates are shorter than this one. The normal curve is also said to be asymptotic. By this we mean that theoretically the tails never touch the base line but extend to infinity in either direction. In actual practice, however, as we have pointed out before, three standard deviations on either side of the curve will include practically all of the cases.

In our educational and psychological work, we assume that certain traits are normally distributed. In actuality, probably no distribution ever takes on the absolute form of the normal distribution. Many of our frequency distributions are very close to the normal one and we assume that we have a normal distribution. To the extent that our distributions differ from the normal one, error enters into our work. The normal curve is important not primarily because *scores* are assumed to be normally distributed but because the *sampling* distributions of various statistics are known or assumed to be normal. Hence its importance is primarily in sampling statistics. This will become apparent when the material in Chapters 11 and 12 is studied.

The normal curve is a mathematical entity and is represented by the following rather awesome formula:

$$y = \frac{N}{\sigma\sqrt{2\Pi}} e^{-\frac{1}{2}(x/\sigma)^2} \tag{7.1}$$

where

y = an ordinate taken at any point on the base line.
Π = 3.1416.
e = 2.7183, the base of the system of natural logarithms.
σ = the standard deviation of the distribution.
x = the deviation of any unit of measurement from the mean.

It will be noted in the above equation that we use the symbol σ instead of s because we are now dealing with population values or parameters rather than sample values. However, most often we use the sample value as an estimation of the parameter value. We have nothing else to use.

Using formula 7.1 and substituting appropriate values of different points along the base line, we can solve for the ordinates of these points and then draw our curve, using the heights of these ordinates. In doing this we should take enough points along the base line to cover adequately the range and also have enough of these points so the curve will tend to be smooth. For example, we could let the first of the points which we select be the mean. For the mean the deviation value, x, is equal to 0. Since x is equal to 0, the exponent of e is 0, and any quantity raised to the 0 power is equal to 1. Then formula 7.1 becomes

$$y = \frac{1}{\sqrt{2\Pi}} = \frac{1}{\sqrt{2(3.1416)}} = \frac{1}{\sqrt{6.2832}} = \frac{1}{2.5066}$$
$$y = .3989$$

Examination of Table II in the Appendix reveals that this value of .3989 is the one recorded for the ordinate of the mean standard score. Other values can then be taken for x, the ordinates computed, and the normal curve drawn using the various y values. Equation 7.1 is based upon a normal curve with a mean of 0, a standard deviation of 1, and an area reduced to unity. It is referred to as the unity normal curve or the normal curve in standard score units. In actual practice we do not have to solve this equation at all. The work has all been done for us and the results are shown in Table II in the Appendix. Here we find the ordinates and other values of the normal curve for various standard deviation units from the mean.

We must comment here upon how the area under the curve is used. We have already noted that it is reduced to unity. In solving problems we assume that our cases are spread equally all over this area. Suppose that we have a problem with 500 cases. In solving the problem we find that .262 of the area of the curve falls above a certain point. To determine the number of cases above this point, we merely multiply our proportion by the number of cases, in this case .262(500). We also use this area of the curve in talking about probability. Suppose that as an illustration we take a raw score which has an equivalent standard score of + 1.88. By referring to the normal probability table in the Appendix (Table II) we note that approximately .03 of the area of the curve falls above this point. We then say that the probability of obtaining a score equal to or higher than this one of + 1.88 is .03 or 3 in 100.

AREAS UNDER THE NORMAL CURVE

In the chapter in which we discussed standard deviations we mentioned some of the relationships between standard deviation units and the normal curve. The most important of these relationships are again repeated in Figure 7.1. To summarize, one standard deviation taken on each side of the mean includes a total area of 68.26 percent of the curve of approximately two-thirds of the cases. In terms of probability, we can state that the chances are about two out of three of a score in any sample falling within the area of one standard deviation on each side of the mean. A second standard deviation measured beyond the first one cuts off 13.59 percent of the area.

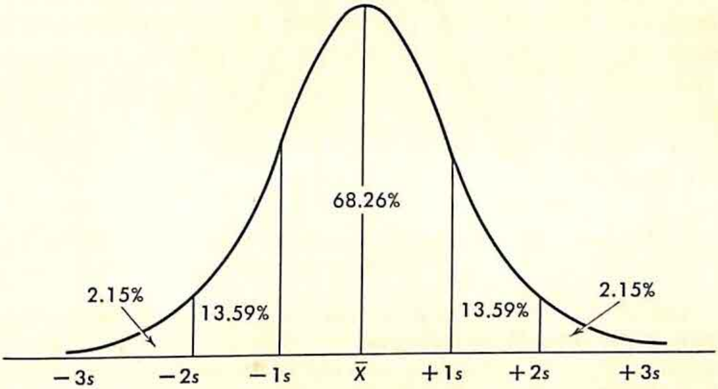


FIGURE 7.1 **Percentages under the normal curve at various standard deviation units from the mean.**

By adding all of the area included by two standard deviation units on both sides of the mean we have accounted for more than 95 percent of the area or cases. Continuing and measuring off another or third standard deviation on each side of the mean, we cut off another piece equal to 2.15 percent of the area. The sum of all of the areas included by these six standard deviation units is equal to 99.74 percent of the total. From this it follows that .26 percent of the cases are beyond three standard deviation units from the mean. This means 26 in 10,000, and dividing this by 2 to distribute these equally on both sides of the mean, we see that on each side we can expect 13 cases in 10,000 to fall beyond three standard deviation units from the mean.

Areas Cut Off Between Different Points. Suppose to begin with we take the problem illustrated in Figure 7.2. We wish to find the proportion of the area or of the number of cases included between the two points + .5s and

+ 1.76*s* units above the mean. The problems are all worked using the normal probability table, Table II in the Appendix. We begin by going to the table and finding the area of the curve cut off between the mean and a point equivalent to a standard score of .5 above the mean. This value appears in the second column of the table and is found to be .1915. Next we continue down the table in the left-hand column until we come to a standard score of 1.76. By looking in column (2) we find that .4608 of the area is included between the mean and this point. Then the area of the curve between these two points is the difference between the two points, .4608 − .1915, which equals .2693. We can then state that approximately 27 percent of the cases fall between these two points or that the probability of a score's falling between these two points is .27.

In the next illustration we shall take two points which are on different sides of the mean. This time we wish to determine what proportion of the normal curve falls between a standard score of −.48 and one of +1.5 (Figure 7.3). There are no values for negative standard scores in Table II. As far as areas go, equal standard scores, whether positive or negative, include equal areas when taken from the mean. From the table we find that a standard score of −.48 cuts off an area of .1844 between it and the mean. A standard score of +1.5 likewise includes .4332 of the area of the curve between it and the mean. The area included between both points is then equal to the sum of these two areas, .1844 + .4332, which is equal to .6176 or approximately 62 percent of the area.

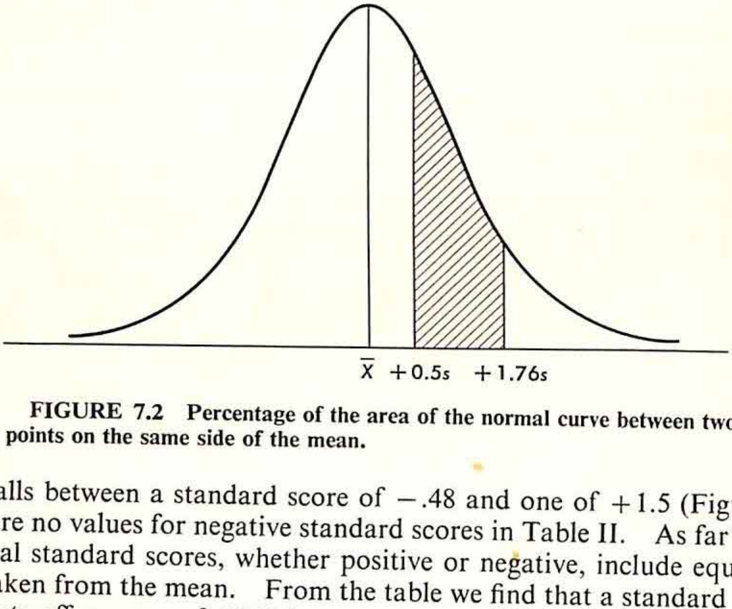


FIGURE 7.2 **Percentage of the area of the normal curve between two points on the same side of the mean.**

Before we go any further, we might examine the other columns of Table II in the Appendix. The third one is labeled the area in the larger portion and the fourth, the area in the smaller portion. Any time we take a point on the base line of a curve and erect a perpendicular at this point, we divide the curve into two areas, a larger one and a smaller one. For any given standard score the sum of these two areas is equal to unity. The last column of the table, the ordinates, gives the size of the ordinates for the various standard scores.

Here is another type of problem that can be solved using these tables. Suppose that we have a distribution of test scores for which the following statistics have been computed: $\bar{X} = 80$, $s = 16$, $N = 510$. We wish to know what percentage of the scores in this distribution fall above a raw score

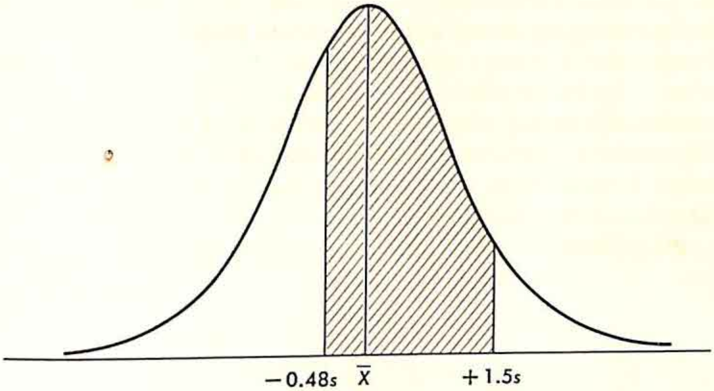


FIGURE 7.3 **Percentage of the area of the normal curve between two points on different sides of the mean.**

of 110 (Figure 7.4). This can, of course, be solved for any score in this distribution. Since the table of the normal curve is in standard units, the first thing to do is to change the raw score of 110 to a standard score as follows:

$$z = \frac{X - \bar{X}}{s} = \frac{110 - 80}{16} = \frac{30}{16} = 1.875$$

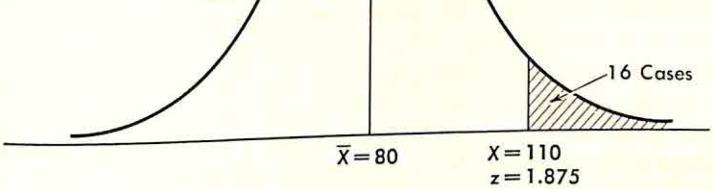


FIGURE 7.4 **Percentage of cases in a normal distribution falling above a certain score.**

Going into the tables we find that this score is halfway between 1.87 and 1.88 so we have to interpolate at this point. This time we are interested in the area in the smaller portion of the curve at a given standard score point. The value from the table is half the distance between .0307 and .0301 which equals .0304. Next we multiply our N by this proportion to find the number of cases above this raw score of 110. We find that $510(.0304) = 15.504$ cases. This in round numbers is approximately 16 cases.

Points Above or Below Which Certain Proportions of the Area Fall. Now we are going to reverse the process we have been using. Suppose that we wish to find that point in the distribution which has 10 percent of the cases below it and 90 percent of the cases above it (Figure 7.5). This point would be the 10th centile. Again we enter the normal probability table and, using the fourth column, the area in the smaller portion, we locate that value closest to .10. In the table we find this to be .1003, and reading to the left we find that a standard score of 1.28 divides the area of the curve into the two proportions as desired. Since this standard score is to the left of the mean it has to have a minus sign in front of it and is correctly written as -1.28. We could have solved this problem just as well by using the third column of the table, the area in the larger portion. We would have gone down this column until we got as close to .90 as possible. This value is .8997 which puts us in the same row as before and the results from there on are similar. If we wanted to find the point in the distribution with 10 percent of the cases above it (C_{90}), the work would be identical to that just completed except that in the end the value of the standard score would be $+1.28$ because this time our point is to the right of the mean.

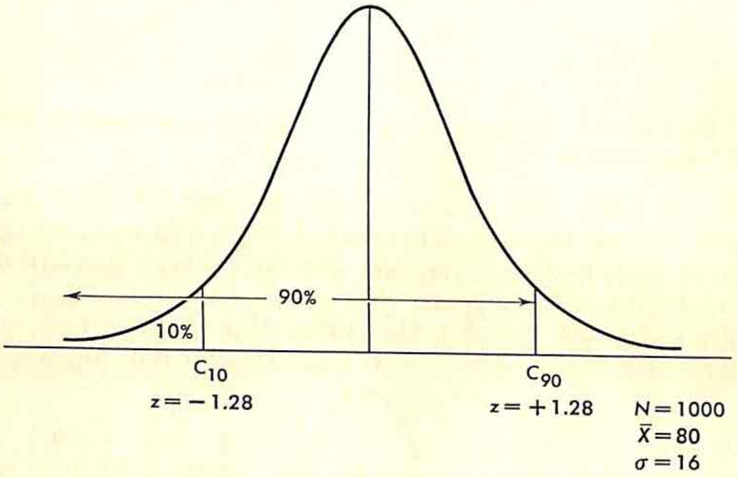


FIGURE 7.5 Points in the normal curve above and below which different percentages of the curve lie.

Setting Up Centile Norms Using the Normal Probability Table. So far we have taken up two methods of setting up a table of centile norms. The first of these was the computation method in which the raw score equivalent for each centile point was computed exactly as was done for the median. Secondly, we constructed an ogive curve and read from this the various centile points. Now we are going to take up a third method using the technique in the previous paragraph. These norms may be referred to as *normalized* centile norms.

To illustrate this process, suppose that we have a distribution with the following statistics and we wish to set up a set of centile norms for the distribution: $\bar{X} = 80$, $s = 16$, $N = 500$. The results of doing this for the various deciles are shown in Table 7.1. Let us take the 9th decile or C_{90} first.

TABLE 7.1. Normalized Centiles for a Set of Data

	z	X
C_{90}	$+1.28$	101
C_{80}	$+.84$	94
C_{70}	$+.525$	89
C_{60}	$+.25$	84
C_{50}	$.00$	80
C_{40}	$-.25$	76
C_{30}	$-.525$	71
C_{20}	$-.84$	66
C_{10}	-1.28	59

NOTE: For these data $\bar{X} = 80$ and $s = 16$.

Going into the table and continuing down in the column of the area in the smaller portion, we find that a standard score of $+1.28$ divides our area with 90 percent of the area below it and 10 percent above it. Our next task is to convert this standard or z-score into a raw score and round it upward. This will be our centile score. The formula for the z-score is as follows:

$$z = \frac{X - \bar{X}}{s}$$

In this case z is known, $\bar{X}$ is known, and so is s. Hence we wish to solve this for X. It will be more convenient if we solve the above equation for X and then substitute into this one.

$$\begin{aligned} z &= \frac{X - \bar{X}}{s} \\ zs &= X - \bar{X} \\ -X &= -zs - \bar{X} \\ X &= zs + \bar{X} \end{aligned}$$

Then for C_{90} in which $z = 1.28$, we substitute

$$\begin{aligned} X &= (1.28)16 + 80 \\ X &= 21 + 80 \\ X &= 101 \end{aligned}$$

We continue in this fashion for the rest of the decile points. It should be noted that after one gets to the mean that the other half of the centile or decile points do not have to be computed as their value is the same distance below the mean as their corresponding point is above the mean. For example, C_{10}, which corresponds to C_{90}, is 21 points below the mean and thus equal to 59.

NORMALIZING A DISTRIBUTION OF SCORES

Next we are going to take up the process by which we normalize a distribution of scores. This is often referred to as setting up the best-fitting normal curve or distribution for a set of data. This best-fitting curve has the same mean and standard deviation as the original data and is based upon the same number of cases.

We shall illustrate this process by using the data in Table 7.2. These data are based upon 160 cases and have a mean of 63.9 and a standard deviation of 22.4. In column (1) appears the frequency distribution and in column (2) the frequencies, which are labeled this time f_0, the observed frequencies. After setting up these two columns, we proceed as follows:

TABLE 7.2. Normalizing a Distribution of Scores

(1)	(2) f_o	(3) Upper Limit	(4) x	(5) z	(6) Proportion Below	(7) Proportion Within	(8) f_e	(9) f_e
110–119	2	119.5	55.6	2.48	.9934	.0141	2.06	2
100–109	8	109.5	45.6	2.04	.9793	.0341	5.46	5
90–99	12	99.5	35.6	1.60	.9452	.0723	11.57	12
80–89	16	89.6	25.6	1.14	.8729	.1180	18.88	19
70–79	22	79.5	15.6	.69	.7549	.1562	24.99	25
60–69	36	69.5	5.6	.25	.5987	.1740	27.88	28
50–59	27	59 5	−4.4	−.19	.4247	.1636	26.18	26
40–49	13	49.5	−14.4	−.64	.2611	.1232	19.71	20
30–39	12	39.5	−24.4	−1.09	.1379	.0761	12.18	12
20–29	8	29.5	−34.4	−1.54	.0618	.0379	6.06	6
10–19	3	19.5	−44.4	−1.98	.0239	.0164	2.62	3
0–9	1	9.5	−54.4	−2.43	.0075	.0075	1.20	1
$\Sigma = 160$						.9934	158.79	159

1. Determine the upper limit of each interval and record these in column (3).
2. Determine the x-values of column (4) by subtracting the mean, 63.9, from each of the upper limits in column (3).
3. Change each of these x-values to a standard score, z, by dividing each by the standard deviation, 22.4.
4. From the normal probability table in the Appendix (Table II), determine the proportion of the area in the normal curve below each of these standard scores. For the bottom one, −2.43, we find from the table that .0075 of the area is below this point. These proportions are recorded in column (6).
5. The values in column (7) are determined as follows. The bottom value of this column would have the same value as the bottom value of column (6), for the proportion within (and below) this interval would be the same as the proportion below the upper limit of the interval 0–9. The proportion within the second interval from the bottom is obtained by subtracting

.0075, the proportion in the bottom interval, from .0239, the proportion below the upper limit of the second interval. This gives a value of .0164. For the proportion within the third interval, we subtract .0239 from .0618. This process is continued until all the values in column (7) are determined. The sum of this column should be close to 1.000 but is actually a little less than this, for there are always cases out in the extreme tails of the normal curve which are not taken into account in this process.

6. The values in column (8) are obtained by multiplying the number of cases, 160, by each of the proportions in column (7). Again these add up to a trifle less than 160. These values are labeled f_e, expected or theoretical frequencies. This distribution, when normalized, has frequency values similar to those noted in column (8). Suppose that we plot these frequencies along with our original observed frequencies. Before we do this, we shall round the f_e values in column (9) to the nearest integer to simplify our plotting. In Figure 7.6, the axes have been prepared in the usual manner for the plotting of a frequency polygon. First the f_0's are plotted and these points connected with a ruler as is the method when plotting a frequency polygon. Then the values in column (9) are plotted and these are connected by means of a smooth curve. In Figure 7.6 we have then the best-fitting normal curve for these data superimposed upon a frequency polygon illustrating the original data.

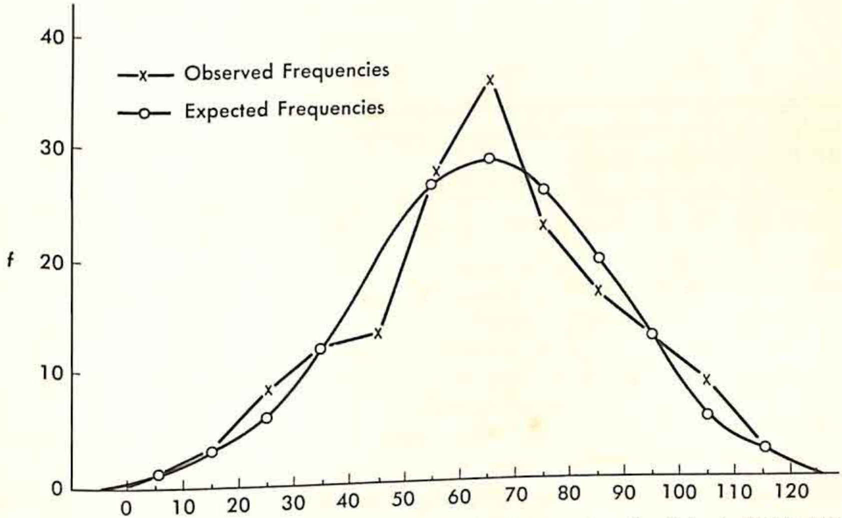


FIGURE 7.6 Frequency polygon and normalized curve for the data in Table 7.2.

The student may by now be wondering when he should normalize a curve or even if there is any justification for performing the act. Later in Chapter 14 on chi square we shall see that one of the uses of chi square is to see if a distribution is normal or not. In order to make a chi square test we must be able to set up the expected frequencies for any set of data. Also, there are times when a distribution of a sample of scores is not normal; yet the research worker has the hunch or knows that the distribution of the trait with which he is concerned is normally distributed in the population. Since he can justify this, he goes ahead and normalizes the data which make up his sample.

Exercises

1. The following terms are associated with the normal curve. Define each term.
    - (a) asymptotic
    - (b) symmetric
    - (c) unimodal
    - (d) mathematical function
    - (e) ordinate
    - (f) smooth
2. Given the following z's find the corresponding area from the mean to the z.

    $.27$
    $-.15$
    3.00
    1.00
    -2.47
    $.80$
    $-.05$
    $-.65$

3. Given an N of 300 and a normally distributed population, how many cases would be expected to exceed each of the z's given in Exercise 2? Then express each of the z's as a centile point.
4. If a frequency polygon were drawn for each of the following populations, which ones do you think would approximate the normal curve?
    - (a) Age (in months) of fourth grade pupils.
    - (b) Income (in dollars) of 40-year-old males.
    - (c) Height of cornstalks (in inches) in a field.
    - (d) Weight (in ounces) of statistics textbooks.
    - (e) Means of an infinite number of samples ($N = 10$) drawn from a normally distributed, infinitely large population.
    - (f) Errors in the estimation of the speed of an earth satellite. (Assume no errors in instrumentation.)
5. Below are the scores of 130 university students on the *Minnesota Paper Form Board Test.*

60–62	2
57–59	4
54–56	8
51–53	10
48–50	18
45–47	16
42–44	14
39–41	14
36–38	10
33–35	18
30–32	14
27–29	2

$N = 130$
$\bar{X} = 42.8$
$s = 8.2$

(a) Find and plot the best-fitting normal curve for these data. On the same axes plot the frequency polygon for the original data.

(b) What percent of the cases would you expect to find between the mean and the following scores for the above distribution? 60, 52, 38, 34, 28.

(c) Find the percentage and number of cases falling between the following pairs of scores: 35 and 45; 50 and 55; 30 and 40; 56 and 60.

(d) How many cases would you expect to find above a raw score of 50? Below a score of 35?

(e) Compute the 95th, 75th, 20th, and 5th centiles for these data.

8

CORRELATION—THE PEARSON r

Correlation is basically a measure of relationship between two or more variables. If we notice some of the common things that we measure, we see that high grades in English tend to be associated with high grades in foreign languages. Both of these tend to be associated with high scores on intelligence tests. In the physical domain, tall people tend to be heavier than short people. The volume of a gas is related to the pressure under which it is kept. In the field of economics, there is a correlation between the price at which products are sold and the amount available for sale. So in all aspects of life, we find that there are relationships of one sort or another. It should be noted here that these relationships do not necessarily imply that one is the cause of the other. This may or may not be the case. In some situations, we find that two variables are related because they are both related to or caused by a third variable.

The size of the Pearson product-moment correlation coefficient varies from $+1$ through 0 to -1. Most correlation coefficients tell us two things. First we have an indication of the magnitude of the relationship. It is worth noting at this point that a correlation of $-.88$ is the same size as one of $+.88$. The sign has nothing to do with the size of the relationship, but it does give information about the direction of the relationship. When two variables are positively related, as one increases, the other also increases. Intelligence test scores are positively related to academic grades. In general, the higher the intelligence test scores, the higher the grades received in school. Other variables are inversely related. By this we mean that as one increases, the other decreases. Think for a moment of the relationship between the speed of an automobile in high gear and the number of miles per gallon obtained from a gallon of gasoline. The faster that one drives, the worse the gasoline mileage. The absence of a relationship is denoted by a correlation coefficient of .00 or thereabouts.

A Perfect Positive Relationship. Figure 8.1 has been constructed to show simultaneously the relationship between two variables. Note how it is set up. On the abscissa are plotted the X-values and on the ordinate, the Y-

values. Each point on the graph stands for an individual's score on both *X* and *Y*. Such a double-entry chart as this is referred to as a scattergram or scatterplot. It should be emphasized that a scatterplot should be set up every time one is computing a correlation coefficient, no matter what computing technique is being used. The importance of this will be discussed later. To return to Figure 8.1, notice that for every increase of two units on the *X*-variable, there is a corresponding increase of one unit on the *Y*-variable. This is true for all pairs of the two variables. When pairs of values like these are plotted, they fall along a straight line, and, when this straight line runs from the lower left of the scattergram to the upper right, we have an example of a perfect positive relationship. The correlation coefficient is equal to +1.00.

X	Y
20	10
18	9
16	8
14	7
12	6
10	5
8	4
6	3
4	2
2	1

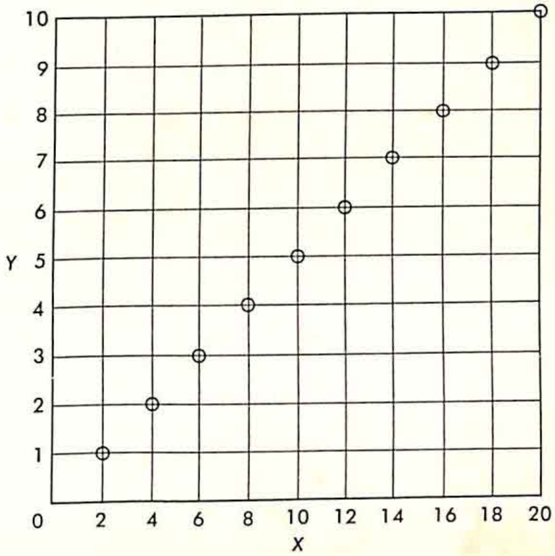


FIGURE 8.1 **A perfect positive relationship.**

A Perfect Negative Relationship. Figure 8.2 is just the opposite of this. Notice that for these data, for every increase of two units on the *X*-axis, there is a corresponding decrease of one unit on the *Y*-axis. Again this relationship is maintained throughout the range. This time our points again fall along a straight line which now runs from the upper left-hand part of the scatterplot to the lower right. This is the example of a perfect negative relationship, a value of −1.00. This means that the individual with the highest score on the *X*-variable had the lowest score on the *Y*-variable. The second highest scorer on the *X* was second lowest on *Y*, until at the bottom, the lowest on *X* was the highest on *Y*.

Other Relationships. In actual life situations we usually have situations in which the relationship is not perfect. Figure 8.3 shows the scatterplot of a very high positive correlation. Notice here that while the tallies do not fall along a straight line, the line is still apparent. In Figure 8.4 we have the illustration of a low negative relationship. On this scatterplot the tallies are scattered more or less all over. There is an indication of the tallies falling in the general direction from the upper left to the lower right. When the

X	Y
20	0
18	1
16	2
14	3
12	4
10	5
8	6
6	7
4	8
2	9

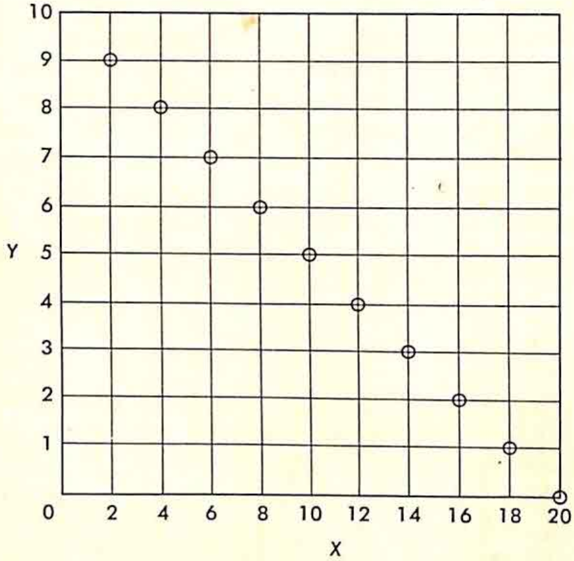


FIGURE 8.2 A perfect negative relationship.

X	Y
20	12
18	16
16	10
15	14
14	12
12	10
12	9
10	8
8	7
5	2

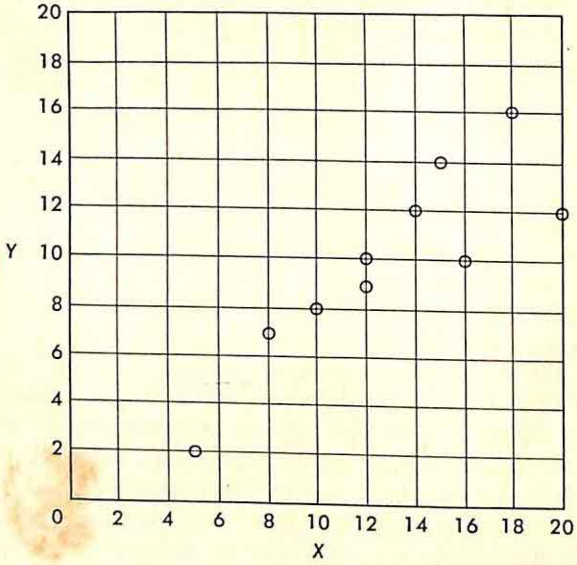


FIGURE 8.3 A high positive relationship (.87).

tallies in the scattergram are spread evenly all over it, we have an example of no relationship.

By making scatterplots, the student can get an indication of the magnitude of the correlation coefficient and also an indication of the sign of the relationship. This, in part, is a rough check on the computational work.

Assumptions Involved in the Computation of r. Before *r* is computed the

X	Y
20	4
18	8
16	1
14	5
12	2
10	10
8	4
6	4
4	10
2	6

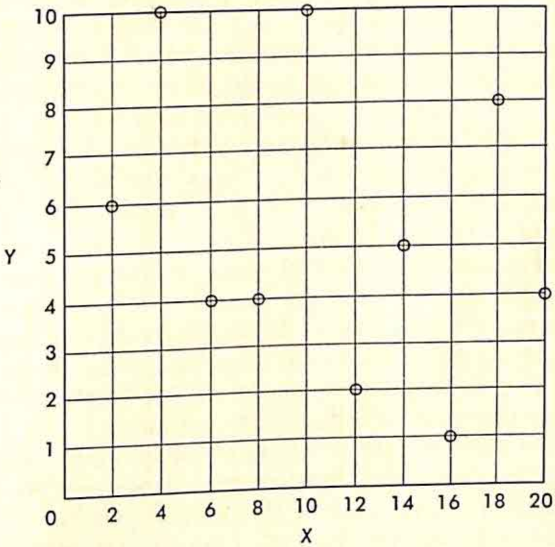


FIGURE 8.4 A low negative relationship (−.31).

X	Y
10	1
9	2
8	3
7	4
6	4.5
5	4.5
4	4
3	3
2	2
1	1

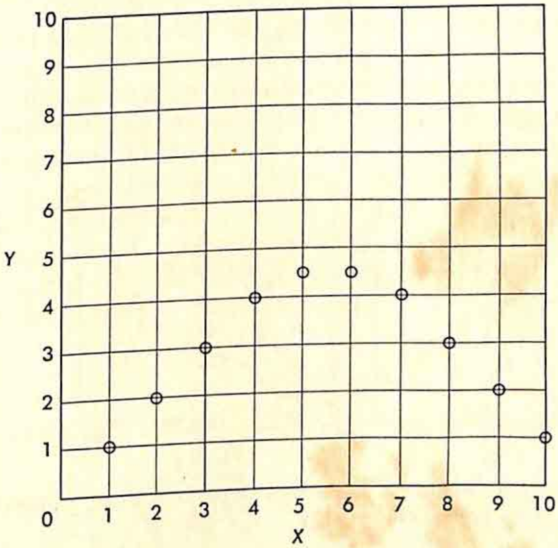


FIGURE 8.5 An example of curvilinear regression.

data should be examined and tested to see if two basic assumptions are met. The first of these is that we have linear regression (also referred to as rectilinear regression). This means that our tally plots on the scattergram tend to fall along a straight line. This is true of the data pictured in Figures 8.1, 8.2, and 8.3. However, now look at Figure 8.5. Notice here that beginning at the lower left, as the Y-variable increases, so does the X-variable; but as X keeps increasing, Y begins to slow down, and further along as X continues to increase, Y starts and continues to decrease. This pattern of relationship is referred to as curvilinear regression. Such relationships are by no means rare. Consider many of our motor skills. As we get older, we are able to run faster. Then comes a period of no improvement, and then as we further increase in age, our running speed decreases. Scores of school children on intelligence tests and personality inventories show the same pattern. The average child in mental ability has fewer personality problems than either the bright child or the dull child.

A more exact description of linear regression is that the means of the columns and rows fall along a straight line. Suppose that you refer to a scattergram similar to that of Table 8.4 but having many more frequencies. Starting at the left it would be possible to find the mean of each of these Y-columns and to plot this mean on the scattergram. If there is linear regression, these means will tend to fall along a straight line. In a similar fashion the mean of each row could be determined and then plotted. There are then two regression lines (except when $r = 1.00$, when they coincide). We shall discuss these lines in the next chapter.

To the extent that one does not meet this assumption of linearity, the size of the computed r is diminished. It is possible that a very high relationship appears to be almost nothing because of the use of the incorrect statistic. When regression is curvilinear, the eta coefficient or the correlation ratio must be used. This is discussed in Chapter 16.

The second assumption is that the data possess homoscedasticity. By this is meant that the standard deviations (or variances) of the arrays (columns and rows) tend to be equal. In Figure 8.6 there are three diagrams. In (*a*) the variance of the columns near the center of the X-distribution is smaller than that of the extreme columns. In diagram (*b*) all of the columns at the left have smaller variances than those of the center and right. Diagram (*c*) represents the situation when both assumptions are met. Note that the tallies tend to fall in about an ellipse built about the regression line.

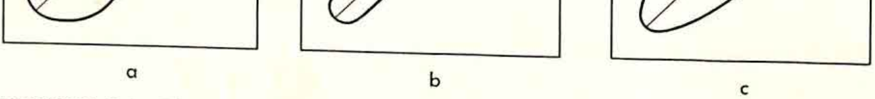


FIGURE 8.6 Figures *a* and *b* lack homoscedasticity. Figure *c* has both homoscedasticity and linear regression.

Computation of the Pearson Product-Moment Correlation Coefficient. Of the various correlation coefficients in current use, the one most frequently encountered is called the Pearson product-moment correlation coefficient, the symbol of which is r. In the pages that follow, we shall present various methods for computing this coefficient.

The Pearson r *from Standard Scores.* Table 8.1 presents the scores of ten individuals on two variables, X and Y (columns (1) and (2)). Beneath each of these is the mean. In column (3) is found the deviation of each of the X-scores from the mean of X, and in column (4) we find the deviation of each of the Y-scores from the mean of Y. Beneath these two columns are the standard deviations of the columns. In columns (5) and (6) are the standard scores for each of the scores in columns (1) and (2). These were obtained by dividing each score in column (3) by s_x (4.34) and each value in column (4) by s_y (3.7). Here we are using the usual formula for z, $z = x/s$. Column (7) is the product of the two z-scores, the product of the values in columns (5) and (6). These are summed, and the Pearson r is obtained by the following formula which describes r as the mean z-score product:

TABLE 8.1. Computation of the Pearson Product-Moment Correlation Coefficient by the Use of Deviations from the Means

(1) X	(2) Y	(3) x	(4) y	(5) z_x	(6) z_y	(7) $z_x z_y$	(8) x^2	(9) y^2	(10) xy
20	12	7	2	1.61	.54	.8694	49	4	14
18	16	5	6	1.15	1.62	1.8637	25	36	30
16	10	3	0	.69	.00	.0000	9	0	0
15	14	2	4	.46	1.08	.4968	4	16	8
14	12	1	2	.23	.54	.1242	1	4	2
12	10	−1	0	−.23	.00	.0000	1	0	0
12	9	−1	−1	−.23	−.27	.0621	1	1	1
10	8	−3	−2	−.69	−.54	.3726	9	4	6
8	7	−5	−3	−1.15	−.81	.9315	25	9	15
5	2	−8	−8	−1.84	−2.16	3.9744	64	64	64
$\Sigma=130$ $\bar{X}=13$	$\Sigma=100$ $\bar{Y}=10$	$s_x=4.34$ $s_y=3.7$				$\Sigma=8.6947$	$\Sigma=188$	$\Sigma=138$	$\Sigma=140$

$$r = \frac{\Sigma z_x z_y}{N} \tag{8.1}$$

Substituting into this formula, we obtain

$$r = \frac{8.6947}{10}$$

$$r = .869 \quad \text{or} \quad .87$$

The Pearson r *from the Deviations from the Means.* Since a standard score is defined as the deviation of a score from its mean divided by its standard deviation, we can write

$$z_x = \frac{x}{s_x} \quad \text{and} \quad z_y = \frac{y}{s_y}$$

Then taking formula 8.1 and substituting for z_x and z_y each of the above and solving, we can reduce the formula for r to

$$r = \frac{\Sigma xy}{Ns_x s_y} \tag{8.2}$$

which further reduces to

$$r = \frac{\Sigma xy}{\sqrt{(\Sigma x^2)(\Sigma y^2)}} \tag{8.3}$$

In columns (8), (9), and (10) of Table 8.1 are the values for x^2, y^2, and xy, and the sum of each column is given at the bottom. Substituting these into formula 8.3, we obtain

$$r = \frac{140}{\sqrt{(188)(138)}}$$

$$r = \frac{140}{\sqrt{25955}} = \frac{140}{161}$$

$$r = .87$$

which is identical with the value obtained previously.

Both of the above methods would be very laborious with an N of any size. We shall pass on next to the two methods most frequently encountered in the computation of r.

The Machine Formula for Computing r *from Raw Scores.* We shall now go from formula 8.3 to the so-called machine formula. We have previously written the sum of the squares as

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{N}$$

$$\Sigma y^2 = \Sigma Y^2 - \frac{(\Sigma Y)^2}{N}$$

TABLE 8.2. The Pearson r Computed from Raw Scores

(1) X	(2) Y	(3) X^2	(4) Y^2	(5) XY
20	12	400	144	240
18	16	324	256	288
16	10	256	100	160
15	14	225	196	210
14	12	196	144	168
12	10	144	100	120
12	9	144	81	108
10	8	100	64	80
8	7	64	49	56
5	2	25	4	10
$\Sigma =$ 130	100	1878	1138	1440

and it follows by analogy that

$$\Sigma xy = \Sigma XY - \frac{(\Sigma X)(\Sigma Y)}{N}$$

When we substitute the above values into formula 8.3, we have

$$r = \frac{N\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{[N\Sigma X^2 - (\Sigma X)^2][N\Sigma Y^2 - (\Sigma Y)^2]}} \tag{8.4}$$

The same data that appear in Table 8.1 have been reproduced in Table 8.2. The original pairs of scores appear in columns (1) and (2). In column (3) are the squares of each of the X-scores and in column (4) are the squares of each of the Y-scores. Column (5) consists of the cross-products, the product of each X times each Y. All five columns are summed, and we enter formula 8.4 as follows:

$$r = \frac{10(1440) - (130)(100)}{\sqrt{[10(1878) - (130)^2][10(1138) - (100)^2]}}$$

$$r = \frac{14400 - 13000}{\sqrt{(18780 - 16900)(11380 - 10000)}}$$

$$r = \frac{1400}{\sqrt{(1880)\,(1380)}}$$

$$r = \frac{1400}{\sqrt{2594400}}$$

$$r = \frac{1400}{1610}$$

$$r = .87$$

which we have obtained twice previously for the same set of data.

If the above formula is used as a hand formula, considerable time is saved by remembering that there is a table of squares in the Appendix. It should also be noted that much time will be saved if the data are coded. Consider the data in Table 8.3. Notice the scores in the first column. They are all large, the smallest being 80 and the highest 114. If each of these scores is first reduced by 80 and the values in the second column by 30 (this would result in one negative score, but would be more convenient than reducing each score by 28, which is the lowest score in this distribution), the correlation coefficient computed from the coded data would be identical to that obtained from the uncoded scores, and the work would be much easier. Even if all of the work is to be done on the calculating machine, many people find that it is time-saving to code the scores in the manner indicated. If automatic calculators are available, it is not necessary to go through all of the steps as shown in Table 8.2 as the machine keeps running sums of the sum of the X's, the sum of the Y's, the sum of the X^2's, the sum of the Y^2's, and the sum of the XY's as each pair of scores is entered. When the last pair has been put into the machine, the final readings of all five values appear. The work in the solving of the formula is also considerably reduced if a calculating machine is available.

The Scattergram Method. The final method to be taken up consists of tallying all of the scores into a scattergram and solving the correlation from that. It has been recommended that the student always plot his data into a scattergram when he is planning to compute a correlation coefficient.

In Table 8.3 are the scores of 35 university students on two art judgment tests, the *Meier Art Judgment Test* and the *Graves Design Judgment Test.* We shall compute the Pearson r between these two sets of scores by the scattergram method.

TABLE 8.3. Scores of Thirty-Five Students on Two Art Judgment Tests

Individual	Meier Art Judgment	Graves Design	Individual	Meier Art Judgment	Graves Design
1	80	61	19	105	86
2	95	28	20	80	63
3	94	74	21	85	31
4	101	46	22	93	57
5	105	44	23	85	70
6	89	38	24	92	43
7	106	72	25	90	70
8	92	41	26	89	54
9	105	49	27	85	51
10	107	69	28	96	58
11	111	82	29	85	63
12	114	76	30	98	73
13	83	39	31	101	71
14	112	64	32	106	76
15	91	77	33	112	76
16	88	50	34	93	59
17	105	55	35	110	71
18	106	59			

In doing this we go through the following steps:

1. Set up two frequency distributions on the scattergram. We shall enter the Meier scores on the Y-axis and the Graves scores on the horizontal or X-axis. These frequency distributions are set up in the usual manner. Notice that the size of the class interval for the data on the Y-axis is 3; whereas that on the other axis is 5. The size of the class intervals of the two distributions does not have to be the same.
2. Next we tally the scores. We take the first pair, which is a score of 80 on the Y-axis and 61 on the X-axis. To enter this we go up the Y-axis until we come to the interval containing 80 and then across to the interval containing 61. We enter our tally mark in the cell where these two class intervals meet. We then take the next pair, 95 and 28, and go up the Y-axis until we come to the interval containing 95 and then over to the interval containing 28. Again we place the tally mark in the cell where the two intervals meet. We continue this process until all of the pairs of scores

are entered. One is very apt to make mistakes in this plotting and it is a good idea, if time is available, to replot the two sets of frequencies and compare results before proceeding with the computations. An inspection of our tally marks leads us to anticipate a moderate size correlation coefficient for these data.

3. Next we go to the bottom of the scatterplot and sum our frequencies, row A. The next three rows should be familiar to the student. Notice that this time the arbitrary reference point for the *x*-variable is taken as the midpoint of the interval 25–29. Rows C and D are then summed.
4. Then we proceed to the right-hand side of the scattergram and complete columns P, Q, R, and S. This time the arbitrary reference point is taken to be the midpoint of the interval 78–80. Sum columns R and S.
5. Column T is new. Notice that it is labeled $x'y'$, the product of the deviation from the two arbitrary reference points. Since deviations are moments, this could be called the product-moment. Hence the name for this correlational technique. Let us see how the $x'y'$ is obtained. Let us go to the row 114–116. In this row there is one tally and this tally is 10 units from the arbitrary reference point for X and 12 units from the reference point for Y and hence has a product-moment of 120. In the next row, since we have three tallies, we have to find the sum of each $x'y'$. The first one is 77 (7 × 11), the second 110 (10 × 11), and the third 121 (11 × 11). The sum of these three is 308. In the next row there is only one tally with an $x'y'$ of 90. In the next row we have $(3 \times 9) + (4 \times 9) + 2(6 \times 9) + (8 \times 9) + (9 \times 9) + (9 \times 10) + (12 \times 9)$ which is equal to 522. In this manner all of the $x'y'$ values are obtained. This column is then summed.
6. The value of r is obtained by solving the following formula for r using coded values:

$$r = \frac{\Sigma x'y' - \dfrac{(\Sigma fx')(\Sigma fy')}{N}}{\sqrt{\left[\Sigma fx'^2 - \left(\dfrac{\Sigma fx'}{N}\right)^2\right]\left[\Sigma fy'^2 - \left(\dfrac{\Sigma fy'}{N}\right)^2\right]}} \tag{8.5}$$

$$r = \frac{1508 - \dfrac{(229)(207)}{35}}{\left[1815 - \dfrac{(229)^2}{35}\right]\left[1632 - \dfrac{(207)^2}{35}\right]}$$

$$r = \frac{1508 - 1354.4}{\sqrt{(1815 - 1498.3)(1632 - 1222.5)}}$$

$$r = \frac{153.6}{\sqrt{(316.7)(409.5)}}$$

$$r = \frac{153.6}{\sqrt{129688.65}}$$

$$r = \frac{153.6}{360}$$

$$r = .427 = .43$$

TABLE 8.4. Calculation of the Pearson r by the Scattergram Method

X-Axis—*Graves Design Judgment Test*

Y-Axis—*Meier Art Judgment Test*

		25–29	30–34	35–39	40–44	45–49	50–54	55–59	60–64	65–69	70–74	75–79	80–84	85–90		P	Q	R	S	T
		0	1	2	3	4	5	6	7	8	9	10	11	12	13	f	y'	fy'	fy'^2	$x'y'$
	13																13			
114–116	12															1	12	12	144	120
111–113	11															3	11	33	363	308
108–110	10															1	10	10	100	90
105–107	9															8	9	72	648	522
102–104	8															0	8	0	0	0
99–101	7															2	7	14	98	91
96–98	6															2	6	12	72	90
93–95	5															4	5	20	100	105
90–92	4															4	4	16	64	100
87–89	3															3	3	9	27	36
84–86	2															4	2	8	16	44
81–83	1															1	1	1	1	2
78–80	0															2	0	0	0	0
(*A*)	f	1	1	2	3	2	3	5	4	1	7	4	1	1		$35 = N$		$\sum = 207$	$\sum = 1632$	$\sum = 1508$
(*B*)	x'	0	1	2	3	4	5	6	7	8	9	10	11	12	13					
(*C*)	fx'	0	1	4	9	8	15	30	28	8	63	40	11	12					$\sum = 229$	
(*D*)	fx'^2	2	1	8	27	32	75	180	196	64	567	400	121	144					$\sum = 1815$	

It is now possible to obtain the means and standard deviations for each of the distributions from the data on the scatterplot. Let us take the variable on the X-axis first.

$$\bar{X} = 27 + \frac{\Sigma fx'}{N}(i) \qquad \Sigma x'^2 = \Sigma fx'^2 - \frac{(\Sigma fx')^2}{N}$$

$$\bar{X} = 27 + \frac{229}{35}(5) \qquad \text{From above this is } 316.7$$

$$\bar{X} = 27 + 32.7 \qquad \Sigma x^2 = i^2(\Sigma x'^2)$$

$$\bar{X} = 59.7 \qquad = 5^2(316.7) = 7917.5$$

$$s_x = \sqrt{\frac{7917.5}{34}} = \sqrt{232.8} = 15.2$$

Similarly, the mean and standard deviations of the Y-variable are found.

$$\bar{Y} = 79 + \frac{207}{35}(3) \qquad \Sigma y^2 = \Sigma y'^2\,(i^2)$$

$$= 79 + 17.7 \qquad \Sigma y^2 = 409.5(9)$$

$$= 96.7 \qquad \Sigma y^2 = 3685.5$$

$$s_y = \sqrt{\frac{3685.5}{34}} = \sqrt{108.4}$$

$$s_y = 10.4$$

Correlation Coefficients and the Range. Anyone who uses the correlation coefficient to any extent soon notices that the size of the correlation coefficient is rather directly related to the range in the two variables being correlated. For example, if one were to go into any fourth grade and correlate the height of the children in that grade with their weights, he would probably end up with a rather low coefficient. However, if he were to take a group of children in grades two through seven, he would find a rather high positive correlation between height and weight. Grades in graduate school when correlated with intelligence test scores are likely to result in low coefficients; whereas there is a moderate sized correlation coefficient between these two variables when children in grades three through eight are so studied. In both of these examples we have two situations, one of which has a more restricted range than the other.

Table 8.5 presents the scores of 16 individuals on two tests. In the two right-hand columns are the ranks of the persons on both. An inspection of the data shows that there is a high positive correlation between the two variables. An individual who is high on one variable tends to be about the same on the other. There is little fluctuation of the ranks within the two columns. Suppose that we examine now the scores and ranks of the top five cases. By doing this we are reducing the range rather considerably. Notice individual 3. On the first test he has rank 3, on the second rank 5. His rank has changed from 3 to 5 and since there were only 5 ranks, this makes a change of 40 percent. When the total test is used, his change is 2 out of a possible 16 or about 12 percent. Thus when we have a restricted group,

TABLE 8.5. Data to Illustrate the Effect of Restricted Range on r

Individual	X	Y	R_x	R_y
1	40	19	1	2
2	38	21	2	1
3	36	16	3	5
4	34	18	4	3
5	30	17	5	4
6	29	14	6	6
7	28	13	7	7
8	26	12	8	8
9	25	10	9	11
10	24	11	10	9
11	22	10	11	11
12	20	8	12	14
13	19	9	13	13
14	16	10	14	11
15	15	4	15	15
16	10	2	16	16

SOURCE: Adapted from Reliability and Confidence. *Test Service Bulletin*, No. 44. New York: Psychological Corporation, 1952.

small changes on one variable may be accompanied with large changes on the other. These large changes lower the size of the r.

That it is the range and not the size of the sample that affects the size of the r can be demonstrated by selecting five other cases. Suppose that we take individuals 1, 5, 9, 13, and 16. If we compute a correlation between these five sets of scores, we would find it to be +1.00.

There are times in test work when correlation coefficients have to be corrected for restricted ranges. Suppose that one considers test work in the Air Force. To get into pilot training, an individual has to pass a rigorous battery of tests. The poor and low scorers are not accepted. Air Cadets are then a select group. After they are selected they are given further training and testing. Some of these drop out or are dropped. Finally we have a more selected group getting commissioned. If we were to use the group of survivors against which to study our selection tests, we would soon find that our correlation coefficients were just about zero. However, there are equations to take care of situations like this, and by using them, we can correct the obtained coefficients for restricted range. These may be found in Guilford (**6**) or Lindquist (**7**).

Exercises

1. For each of the following pairs of variables tell whether you would expect a strong or weak relationship, a linear or curvilinear, a positive or negative relationship.
   (a) The number of schools in a rural area and the number of students living in that area.

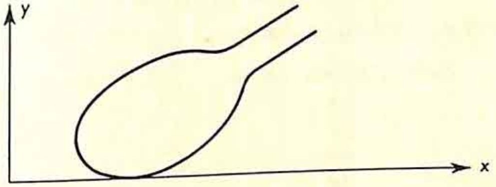
(b) The number of Negro residents of a state and the number of segregated schools.

(c) The height and age of a given species of trees.

(d) The social status and the proportion of income spent on food of a group of families.

(e) The birthrate and degree of urbanization in a culture.

2. Why is the Pearson r reduced when the assumption of linearity is not met?
3. The figure below represents the shape of a scatterplot. What assumption has not been met?



4. A class of 30 students takes two forms of the same test.

$$\Sigma z_x z_y = 27.585$$

What is the correlation between the two forms?

5. A group of 16 students take a mental ability test and an English achievement test. Their scores were as follows:

English	*Mental Ability*
149	44
157	44
163	43
173	46
203	54
141	42
202	44
158	43
176	45
188	53
163	39
192	47
139	37
169	43
162	40
161	46

Compute r by using formula 8.3.

6. Ten people are asked to guess the weight of a package to the nearest pound. Two weeks later they are asked to guess the weight of the same package; this time they are told it contains something worth $1500.

First Guess	*Second Guess*
7	9
7	9
7	9
8	9
9	10
6	9
7	9
7	9
8	9
9	10

Using the raw score formula (8.4), compute the r.

7. Use the scattergram method to compute r for these data.

Sales Attitude Score	*Sales Index*
48	22
48	19
47	20
46	20
46	17
43	21
42	21
42	19
41	17
40	15
39	18
38	15
38	15
37	20
37	17
35	19
34	15
34	14
33	20
33	13
32	15
32	12
32	11
31	17
30	16
29	15
29	11
28	16
27	16
27	13
27	12
26	12
25	15
25	9
23	9
22	13

Sales Attitude Score	*Sales Index*
21	9
20	11
18	11
17	10
15	8

8. The following scores were made by 30 individuals:

	Minn. Clerical Numbers	*Minn. Clerical Names*	*Minn. Paper Form Board*	*Otis Quick Scoring Mental Ability*
1	68	74	32	42
2	169	167	45	54
3	87	98	45	62
4	93	76	34	45
5	87	65	24	38
6	77	74	27	32
7	154	145	34	36
8	99	69	38	42
9	87	98	22	37
10	110	98	64	60
11	107	104	47	66
12	119	87	45	45
13	67	78	24	42
14	78	100	20	46
15	186	169	57	55
16	113	97	45	47
17	145	159	40	40
18	65	78	20	33
19	106	106	34	48
20	98	105	34	52
21	76	96	49	48
22	119	125	60	54
23	100	109	24	44
24	118	109	36	56
25	154	174	32	42
26	87	88	23	39
27	109	101	41	47
28	88	108	31	48
29	67	76	18	42
30	89	100	47	68

Compute the correlation among some or all of the above scores using the method directed by the instructor.

9

LINEAR REGRESSION

In the previous chapter we noted that one of the basic assumptions to be met before a Pearson product-moment correlation is computed is that there be a linear relationship between the two variables being correlated. In this chapter we shall examine in more detail this phenomenon of regression and the use that is made of regression in predicting scores on one variable from scores made on another. A common example of the use of the regression equations is the prediction of grades in a particular academic subject or grade averages from scores made on an intelligence test. In this case the grades are referred to as the dependent variable and the intelligence scores as the independent variable. By the use of a regression equation we can predict scores on the dependent variable from those of the independent variable. Usually the independent variable is denoted by the symbol X and the dependent one by Y.

The Equation of the Straight Line. We shall start by discussing the mathematical equation for a straight line. In its simplest form this is

$$Y = a + bX \tag{9.1}$$

First we must find out what a and b stand for. Suppose to do this we write the equation as $Y = 4 + 2X$. We are now able to substitute any value that we wish for X and solve for the corresponding value of Y. A few of these values are shown below.

X	Y
0	4
1	6
2	8
3	10
4	12

Sets of values like these can be plotted, and this is illustrated in Figure 9.1. Since we are dealing with a straight line and since it takes only two points to determine a straight line, only two sets of these points need to be plotted. Let us examine this line in detail now. Notice that for an increase of one point on the X-variable there is an increase of two points on the Y-variable. The two in this equation is the b-coefficient of equation 9.1. The b-coefficient gives the relationship between the change in Y in reference to change in X. This ratio of change in one variable to change in another is referred to as the *slope* of the line. In Figure 9.1 another line has been drawn, $Y = 1 + 2X$. This runs parallel to the first line because it has the same slope. Theoretically an infinite number of such lines with a slope of 2 can be drawn on these axes. The slope of both of these lines is positive as the lines run from the lower left to the upper right on the graph. As was noted previously a line located like this one is an indication of a positive relationship between the two variables.

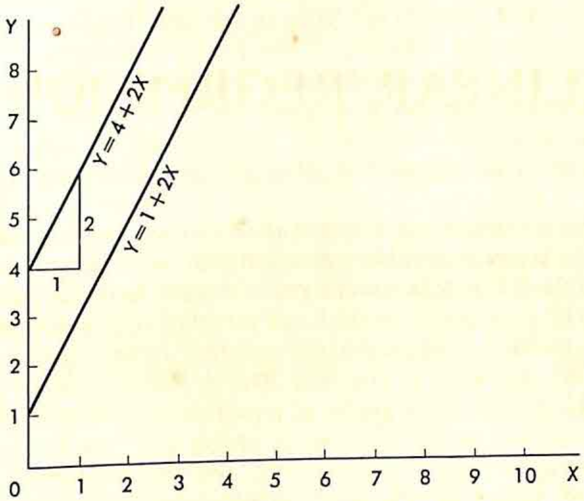


FIGURE 9.1 Two lines with the same slope.

Now examine Figure 9.2. Here we have the equation $Y = 4 - .5X$. Solving for values of Y we have:

X	Y
0	4
1	3.5
2	3
3	2.5
4	2
8	0

This time as X increases from 0 to 1, the corresponding change in Y is from 4 to 3.5. The ratio of the change in Y relative to the change in X is for this example .5. But as one variable is increasing, the other is decreasing, and for this case the b-coefficient is $-.5$.

As was noted above, there is an infinite number of lines of the same slope which can be drawn on the same set of axes. Two such lines are shown in Figure 9.1. Notice that one of these lines crosses the Y-axis at 4 and the other at 1. These values of 4 and 1 are the a-coefficients for the two straight lines. Hence, we might define the *a*-coefficient as the *Y*-intercept, that point where the straight line crosses the *Y*-axis. The *a*-coefficient identifies one of this infinite number of lines of the same slope.

The straight line used in regression analysis is written on a slightly altered form from that given above:

$$Y' = a + bX \tag{9.2}$$

where Y' is read as the predicted value of Y. Y' is not usually the same as Y, for the score that is predicted from this equation will not be exactly the same as the one actually obtained. The values of the predicted Y's will in general be closer to $\bar{Y}$ than are the values of the observed Y's. Because of this relationship, this phenomenon is referred to as regression.

Obtaining the a- and b- Coefficients. In obtaining the values of a and b, a method referred to as the method of least squares is used. The errors of prediction about the line thus obtained are at a minimum. Such a line is referred to as the line of best fit for a set of data. Any values for a and b other than the ones chosen would result in larger errors of prediction. In other words $Y' - Y$-values are at a minimum when this line of best fit is used. The use of the calculus is necessary for obtaining this line of best fit and will not be gone into here.

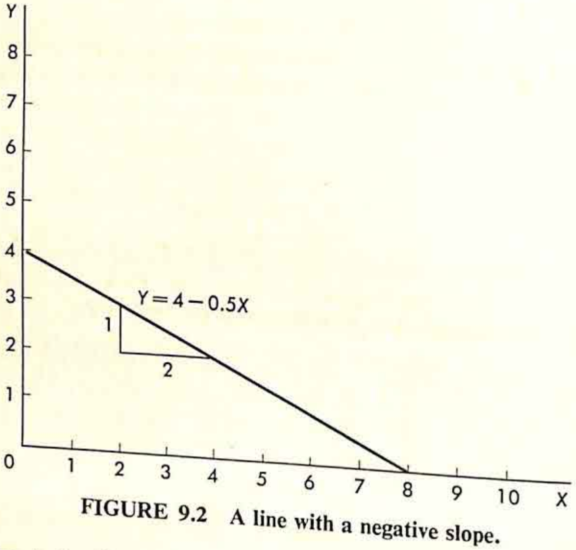


FIGURE 9.2 **A line with a negative slope.**

The formula for the *b*-coefficient for predicting *Y* from *X* is written

$$b_{yx} = \frac{\Sigma XY - \dfrac{(\Sigma X)(\Sigma Y)}{N}}{\Sigma X^2 - \dfrac{(\Sigma X)^2}{N}} \tag{9.3}$$

where N = the number of pairs.

Since we have already noted that

$$\Sigma X^2 - \frac{(\Sigma X)^2}{N} = \Sigma x^2$$

and that

$$\Sigma XY - \frac{(\Sigma X)(\Sigma Y)}{N} = \Sigma xy$$

the *b*-coefficient for predicting *Y* from *X* may be written as follows:

$$b_{yx} = \frac{\Sigma xy}{\Sigma x^2} \tag{9.4}$$

The *a*-coefficient is obtained from the following equation:

$$a_{yx} = \bar{Y} - \bar{X}(b_{yx}) \tag{9.5}$$

where $\bar{X}$ and $\bar{Y}$ are the means of the two variables and b_{yx} is the *b*-coefficient used in predicting *Y* from *X*.

Since there are two regression lines except when there is a perfect relationship between two variables, it follows that there is another set of *a*- and *b*-coefficients, this time those used in predicting *X* from *Y*. These would be written as follows:

$$b_{xy} = \frac{\Sigma xy}{\Sigma y^2} \tag{9.6}$$

$$a_{xy} = \bar{X} - \bar{Y}(b_{xy}) \tag{9.7}$$

An Actual Problem. We shall now illustrate the calculation of the regression lines by using the data in Table 8.4 in the previous chapter. In that problem we ran a correlation between scores on the *Meier Art Judgment Test* and the *Graves Design Judgment Test* and found this correlation coefficient to be .427. Also for the same data we found the following:

$$\bar{X} = 59.7 \qquad \bar{Y} = 96.7$$
$$\Sigma x^2 = 7915.5 \qquad \Sigma y^2 = 3685.5$$
$$N = 43$$

We also need Σxy which is obtained as follows:

$$\Sigma xy = \left[\Sigma x'y' - \frac{(\Sigma x')(\Sigma y')}{N}\right](i_y)(i_x)$$

The student will note that this is the numerator of the equation for the Pearson *r* by the scattergram method multiplied by the size of the two class intervals.

$$\Sigma xy = (153.6)(3)(5) = 153.6(15) = 2304$$

First we find the *b*-coefficient

$$b_{yx} = \frac{\Sigma xy}{\Sigma x^2}$$

$$b_{yx} = \frac{2304}{7915.5}$$

$$b_{yx} = .291$$

We now have the information necessary for the solution of the a-coefficient

$$a_{yx} = \bar{Y} - \bar{X}(b_{yx})$$
$$a_{yx} = 96.7 - (59.7)(.291)$$
$$a_{yx} = 96.7 - 17.4$$
$$a_{yx} = 79.3$$

Then substituting in the equation for a straight line, $Y' = a + bX$, we have $Y' = 79.3 + .291X$ which is the regression equation for the Y-variable on X for these data.

We shall next plot this line. Since two points determine a straight line, we can solve the above regression equation for a couple of points and plot these as shown in Figure 9.3. Suppose that we take three values of X and by substituting these values in the regression equation obtain the predicted values of Y.

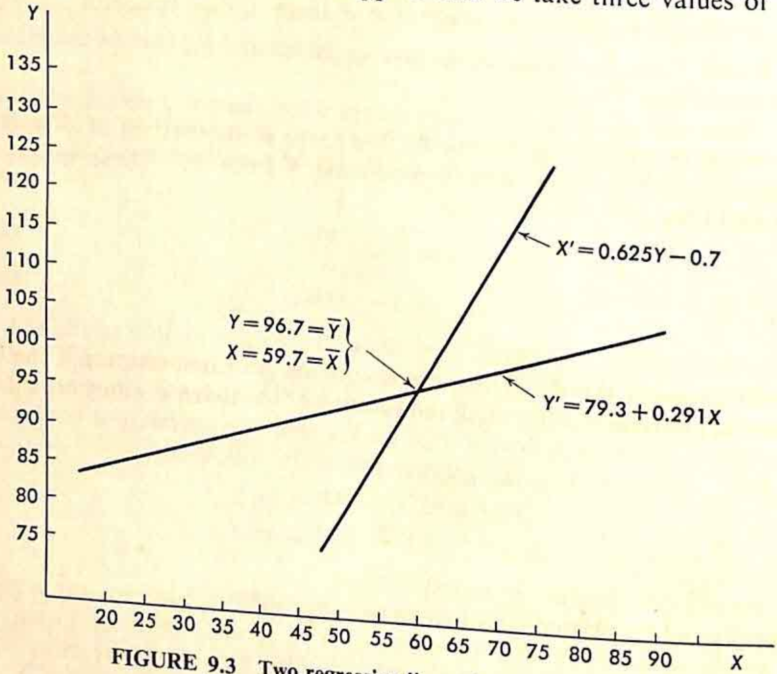


FIGURE 9.3 Two regression lines of the data in Table 8.4.

X	Y'
20	85.1
40	90.9
60	97.8

Plotting these three sets of values on Figure 9.3 results in the regression line shown here.

As we have already noted, there are two regression lines unless $r = 1.00$. We shall now proceed to determine the regression line for predicting X from Y.

First the *b*-coefficient:

$$b_{xy} = \frac{\Sigma xy}{\Sigma y^2}$$

$$b_{xy} = \frac{2304}{3685.5}$$

$$b_{xy} = .625$$

Then the *a*-coefficient:

$$a_{xy} = \bar{X} - (\bar{Y})(b_{xy})$$
$$a_{xy} = 59.7 - (96.7)(.625)$$
$$a_{xy} = -.7$$

The regression equation for predicting X from Y then becomes

$$X' = b_{xy}Y + a_{xy}$$
$$X' = .625Y - .7$$

Let us again obtain three sets of points and plot this line on the same figure as the other. Substituting in the equation for various values of Y, we obtain and then plot the following:

Y	X'
120	75
100	61.8
80	49.3

Checks upon the Work. There are several ways of determining if the calculations are correct. First of all the product of the two *b*-coefficients should equal r^2

$$(b_{yx})(b_{xy}) = r^2$$
$$(.291)(.625) = (.427)^2$$
$$.1819 = .1823$$

which checks disregarding rounding errors.

Secondly, when plotted, the two regression lines should cross at a point equal to the mean of X and the mean of Y. An inspection of Figure 9.3 shows that this is so. This is the same as saying that when the mean of X is substituted in the equation for predicting Y, the predicted value of Y will be the mean of Y.

Substituting for X, the mean of X, 59.7, we obtain

$$Y' = 79.3 + (.291)(59.7)$$
$$Y' = 79.3 + 17.4 = 96.7$$

which is the mean of Y. When the mean of Y is substituted in the other regression equation, the predicted value of X will be the mean of X.

The Standard Error of Estimate. It has already been stated that rarely will our obtained Y-scores be identical with the predicted Y-scores. There is an error in all of our predictions and the extent of this error is measured by a statistic known as the standard error of estimate. When the size of the correlation coefficient between two variables is high, the size of the standard error of estimate is small, and conversely, when the relationship between two variables is low, the size of the standard error of estimate is large. If we have a perfect relationship between X and Y, our obtained Y's will be exactly the same as the predicted ones. This is saying that all of the Y'-values will fall on the regression line. Since there is no variation from this line for the various values of X, there is no standard error of estimate. When we have the opposite situation, the situation where there is no relationship between the two variables, we have a regression line in which the slope is 0. This regression line takes the form of a straight line parallel to the X-axis. Solving the equation for a straight line when b is equal to 0 results in the equation $Y' = a$. When b is equal to 0, $a = \bar{Y}$. Hence, when there is no relationship between two variables, the predicted value of Y is equal to the mean of Y, $Y' = \bar{Y}$. When this situation prevails the size of the standard error of estimate is at a maximum. In this case it is the equivalent of the standard deviation of the Y-variable. The size of the standard error of estimate ranges then from 0 to the size of the standard deviation of the dependent variable (Y). Most of the situations that are encountered fall in between these two extremes.

Since each obtained score differs from the expected or predicted score, the discrepancy between each set of these can be obtained, squared, and then summed. Dividing this by $N - 2$ and then taking the square root results in the standard error of estimate.

$$s_{yx} = \sqrt{\frac{\Sigma(Y - Y')^2}{N - 2}} \tag{9.8}$$

A more useful formula is

$$s_{xy} = \sqrt{\frac{\Sigma y^2 - \dfrac{(\Sigma xy)^2}{\Sigma x^2}}{N - 2}} \tag{9.9}$$

We shall now find the standard error of estimate for predicting X from Y for the data of Table 9.4. Substituting in formula 9.9, we obtain

$$s_{yx} = \sqrt{\frac{3685.5 - \dfrac{(2304)^2}{7915.5}}{35 - 2}}$$

$$s_{yx} = \sqrt{\frac{3685.5 - 670.6}{33}}$$

$$s_{yx} = \sqrt{\frac{3014.7}{33}}$$

$$s_{yx} = \sqrt{91.35}$$

$$s_{yx} = 9.55$$

In a similar fashion the standard error of estimate when predicting X from Y can be obtained.

There is an alternate formula for the standard error of estimate often found in measurement books

$$s_{yx} = s_y \sqrt{1 - r_{xy}^2} \tag{9.10}$$

Solving this for these data, we have

$$s_{yx} = 10.4 \sqrt{1 - (.427)^2}$$

$$s_{yx} = 10.4 \sqrt{.817671}$$

$$s_{yx} = 10.4\,(.904)$$

$$s_{yx} = 9.40$$

These two formulas do not give identical results, but for all practical purposes these are the same.

Interpretation of the Standard Error of Estimate. Let us take the value of 9.55 rounded to 9.6 for the interpretation of this statistic. In predicting Y from X we can say that for any given value of X, the chances are two out of three that the observed cases will fall within a band which is the predicted value of Y plus and minus one standard error of estimate. When X is equal to 20, we would expect to find two-thirds of the observed cases between $85.1(Y') \pm 9.6$ or between 75.5 and 94.7. This is shown on Figure 9.4. Since we assume that the standard error of estimate is constant throughout the range, we could measure off 9.6 units on each side of the regression line for any other value of X and then draw two lines parallel to the regression line. Two-thirds of all of the observed scores should fall between these two outer parallel lines. One could set off two lines within which 95 percent of the observed cases should fall by measuring two standard errors of estimate on each side of the regression line.

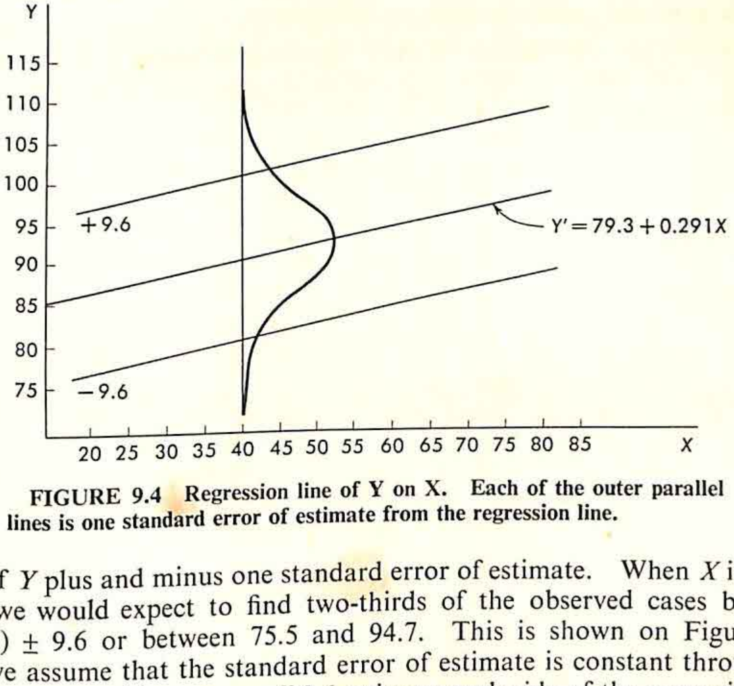


FIGURE 9.4 Regression line of Y on X. Each of the outer parallel lines is one standard error of estimate from the regression line.

Exercises

1. Solve for Y' in each of the following:

(a) $X = 5$ $a = -3$ $b = 2$
(b) $X = -10$ $a = 4$ $b = -2$
(c) $X = -3$ $a = 2$ $b = 3$
(d) $X = 7$ $a = 50$ $b = .125$

2. Since we have a single correlation coefficient for two variables, why do we have two regression lines? How might a practical situation call for one rather than another of the two lines?

3. Given the following data, what would Y' corresponding to an X of 20 be?

X	Y
12	53
23	120
14	61
16	82
15	73

4. What is the standard error of estimate in Exercise 3?

5. A small college is doing research in their entrance testing. A mathematics achievement test is given as part of the entrance battery. Scores on this test and mathematics grades are given for 20 freshmen.

Mathematics Test	*Mathematics Grade*
120	90
112	78
87	61
42	28
56	48
99	71
22	18
50	55
73	81
11	18
63	50
132	96
85	81
93	78
47	45
77	63
61	42
88	73
47	21
75	52

(a) Plot a scattergram for these data.
(b) Compute the a- and b-coefficients for predicting grades from test scores and vice versa.
(c) Construct the regression lines on the scattergram.
(d) Compute the standard error of estimate for each of the lines and plot them on the scattergram.
(e) If 70 is the minimum passing grade, at which test score should candidates be denied admission to this course?
(f) Do you think the math test is doing a good job?

10

THE BINOMIAL DISTRIBUTION AND PROBABILITY

In previous chapters we have talked about probability and used the term in interpreting various statistics. In this chapter we shall consider the laws of probability, the binomial expansion and its relationship to probability, and some applications of this binomial to actual situations.

It is easier to talk about probability than it is to define it. For example, the doctor makes a prognosis about a patient's future recovery, the meteorologist forecasts the weather of the next day, the agriculturist predicts the size of next year's winter wheat crop—each man is making use of a probability statement. The doctor says that if you take this medicine, if you do as I tell you, etc., you will be better in a couple of days. Granted that the doctor does not say that the chances are two out of three that a patient will recover, but the physician can never say that the chances are absolutely perfect that a patient will recover. When we think of it, we find that there are very few things in life about which we can be absolutely certain. One of these is that each of us will someday die. We can say that the probability of the occurrence of this is 1. On the other extreme, we can find some other situations, the probability of whose occurrence is 0. For example, the probability of the reader's swimming the Pacific Ocean to Hawaii or in the course of his lifetime taking a space ship to a remote star is in both of these cases zero.

Most things in life, however, have probability values of their occurrence somewhere between 1 and 0. In making many predictions of outcomes, such as those of the physician or the weather forecaster, a numerical value is rarely assigned to the probability statement. In our work we shall endeavor to be more specific. Let us begin with a familiar object, a one-cent piece. If we take a new coin which in no way has been mutilated and toss it, we can state that the probability of obtaining a head when we toss this coin is one out of two. We write this as $p = {}^1/_2$ or .5. It is also true that the probability of obtaining a tail is ${}^1/_2$ or .5. In this situation, only one of these two outcomes can occur. The toss will result in a head or in a tail coming up. We are not interested in cases in which the coin lands in a crack or against an object and stands on edge. Notice that the sum of the two probabilities is equal to 1. We designate the probability of an event's occurring by use of the symbol p and the probability of its not occurring by the symbol q. The sum of $p + q$ is always equal to one. As an additional example let us take a die (one of a pair of dice) and throw it into the air. If this is an accurate and well-made die, the probability of any single side coming up is $^1/_6$ or .167. That is, when we toss a die, the chance of throwing a six-spot is one in six. There are five in six chances of some other number appearing on the upturned face. In this case, $p = {}^1/_6$ and $q = {}^5/_6$, and again $p + q = 1$. In test work, the probability of a student's getting an answer correct when he knows nothing about the item is, for true-false items, $^1/_2$ or .5. If we have a five-response multiple-choice item, the probability of getting the correct answer by chance is $^1/_5$ or .2.

What then is this probability that we have been talking about? It seems that it is related to the laws of chance. When chance operates, we can make statements about the expected outcomes. These statements can be expressed in terms of probability values. Probability, then, is the theoretical or expected frequency when the laws of chance are operating.

LAWS OF PROBABILITY

To illustrate these, let us return to the tossing of a die. Suppose we toss a die, and this time, suppose that we are interested in the appearance of a three-spot on the upturned face. The probability of a three-spot showing is $^1/_6$. If we toss the same die a second time, the probability of again getting a three-spot is still one in six. What happens the second time is in no way affected by what happened the first time. We say that events like this are mutually exclusive. Even if we tossed ten three-spots in a row, the probability of obtaining a three-spot on the eleventh toss is still one in six. Suppose instead of tossing one die, we toss two dice. What is the probability of their both being three-spots? The probability of one being a three-spot is $^1/_6$ and the probability of the other being a three-spot is also $^1/_6$. The probability of their both being three-spots is $^1/_{36}$. This illustrates the first law of probability. *The probability of the simultaneous or successive occurrence of two or more independent events is equal to the product of their separate probabilities.*

Now let us look at the outcome of the die tossing in another manner. The probability of any given side appearing is one in six. Suppose that we ask, what is the probability of getting either a three-spot or a four-spot. Since the probability of the occurrence of either of these is $^1/_6$, the chances that any given toss of a die will be either a three-spot or a four-spot is $^1/_3$, the sum of the probability of each. This illustrates a second law of probability. *The probability of the occurrence of any of two (or more) mutually exclusive events is equal to the sum of their separate probabilities.*

THE BINOMIAL EXPANSION

If we toss two cents at once, three possibilities may occur. We may find that two heads have come up, a head and a tail, or that both cents show tails. In terms of symbols this may be expressed like this:

First Cent	*Second Cent*
H	H
H	T
T	H
T	T

From this we see that there are actually four possible outcomes, but two of them are the same in the end, for a head and a tail may be produced in two different ways. Since there are four possible outcomes, we can write that the probability of getting two heads is $^1/_4$, the probability of getting a head and a tail is $^1/_2$, and the probability of obtaining both tails is $^1/_4$. The sum of all these probabilities is again equal to one.

The results obtained above are what we would obtain by squaring a binomial. It may be recalled that the square of the binomial $(x + y)$ is equal to $(x^2 + 2xy + y^2)$. In the case of tossing two coins x is the head or H and y the tail or T. If we multiply $(H + T)$ by itself, we obtain $HH + 2HT + TT$.

When we have only two coins, it is fairly easy to see what the probability will be for the various outcomes. But as the number of coins increases, we have to find more efficient ways for obtaining our probability values. This is where we use the binomial theorem, which in its general form looks like this:

$$(p + q)^n = p^n + \frac{n}{1} p^{(n-1)}q + \frac{n(n-1)}{(1)(2)} p^{(n-2)}q^2 + \frac{n(n-1)(n-2)}{(1)(2)(3)} p^{(n-3)}q^3 + \frac{n(n-1)(n-2)(n-3)}{(1)(2)(3)(4)} p^{(n-4)}q^4 + \dots + q^n \tag{10.1}$$

Suppose that we have eight cents and that we toss them simultaneously. Let us see how we can use the above equation. In this problem the probability of getting heads on any one coin, p, is $^1/_2$ and, n, the number of coins, is 8.

Then by substitution

$$\left(\frac{1}{2} + \frac{1}{2}\right)^8 = \left(\frac{1}{2}\right)^8 + \frac{8}{1}\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right) + \frac{8(7)}{1(2)}\left(\frac{1}{2}\right)^6\left(\frac{1}{2}\right)^2 + \frac{8(7)(6)}{1(2)(3)}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^3 + \frac{8(7)(6)(5)}{1(2)(3)(4)}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^4 + \frac{8(7)(6)(5)(4)}{1(2)(3)(4)(5)}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^5 + \frac{8(7)(6)(5)(4)(3)}{1(2)(3)(4)(5)(6)}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^6 + \frac{8(7)(6)(5)(4)(3)(2)}{1(2)(3)(4)(5)(6)(7)}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^7 + \left(\frac{1}{2}\right)^8$$

Simplifying, this becomes

$$\left(\frac{1}{2} + \frac{1}{2}\right)^8 = \frac{1}{256} + \frac{8}{256} + \frac{28}{256} + \frac{56}{256} + \frac{70}{256} + \frac{56}{256} + \frac{28}{256} + \frac{8}{256} + \frac{1}{256}$$

Several things should be noted from the above. In the first place, the sum of all of the values in the numerators must equal the value of the denominator, for the sum of all of the separate probabilities must equal 1. Secondly, after the middle term was reached, each of the other terms repeats one of the earlier ones in descending order of size. Thirdly, in a problem where $p = {}^1/_2$, the denominators are all the same. In this case they were all equal to $({}^1/_2)^8$. Hence after the first one was computed, there was no point in multiplying the rest of the p and q terms. And fourthly, note that there is one more term than n; in this case we have 9 terms in our expansion. Suppose we summarize our results in columnar form, showing the probabilities for the various combinations of heads and tails, as in Table 10.1. From this we see that the probability of getting all heads is .004, about 4 chances in 1000. The probability of getting either 8 heads or 8 tails is .008, the sum of the two separate probabilities. We can also answer other questions such as what is the probability of getting at least 6 heads. The probability of this is the sum of the probabilities for 6 heads, 7 heads, and 8 heads—.109 + .031 + .004 which equals .144 or 144 in 1000.

TABLE 10.1. Probabilities of Obtaining Various Combinations of Heads and Tails When Eight Coins Are Tossed

H	T	p		p
8	0	$^1/_{256}$		.004
7	1	$^8/_{256}$		.031
6	2	$^{28}/_{256}$		.109
5	3	$^{56}/_{256}$		.219
4	4	$^{70}/_{256}$		.273
3	5	$^{56}/_{256}$		.219
2	6	$^{28}/_{256}$		.109
1	7	$^8/_{256}$		.031
0	8	$^1/_{256}$		.004
		$^{256}/_{256}$	=	1.000

Instead of going through all of the work of expanding the binomial as shown above, considerable time may be saved by the use of the so-called Pascal's triangle which is shown in Table 10.2. In the problem used with the binomial we had 8 coins; n is equal to 8. Going to Table 10.2, we find that for $n = 8$ the same values are there as we obtained for the numerators of our probabilities (Table 10.1) and the column at the right gives the denominator of the probability values, which in this case is 256. Notice how the triangle is constructed. A triangle has been drawn upon the triangle. When the 6 and 15 at the top of this small triangle are added, they produce the value 21, which is one of the terms in the row for $n = 7$. By combining all of the terms in row $n = 6$ in this fashion, all of the values in row $n = 7$ are obtained. In this manner the triangle may be expanded indefinitely. Suppose to illustrate this we compute the values for the next row, that is, when $n = 10$. The first term would be 1; then the second term,

TABLE 10.2. Pascal's Triangle

n	Binomial Coefficients	Denominator of p
1	1 1	2
2	1 2 1	4
3	1 3 3 1	8
4	1 4 6 4 1	16
5	1 5 10 10 5 1	32
6	1 6 15 20 15 6 1	64
7	1 7 21 35 35 21 7 1	128
8	1 8 28 56 70 56 28 8 1	256
9	1 9 36 84 126 126 84 36 9 1	512

1 + 9 or 10; the third term, 9 + 36 or 45; the fourth term, 36 + 84, or 120; the fifth term, 184 + 126, or 210; the middle term, 126 + 126, or 252; and the rest of the terms would be the same as the ones already computed in reverse order.

The Binomial and the Normal Distribution. In Figure 10.1 is shown a frequency polygon representing $(p + q)$: when $p = .5$ and $n = 6$. As we

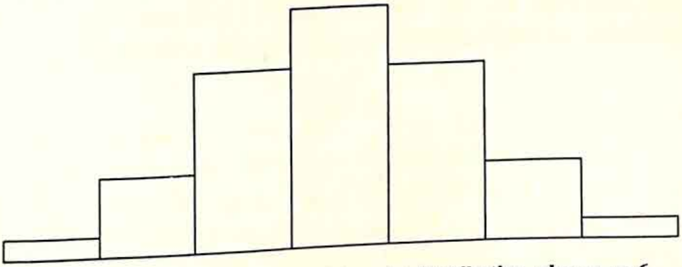


FIGURE 10.1 **Curve of the binomial distribution when $n = 6$.**

might expect, there are seven columns, each in proportion to its probability. Suppose now that instead of tossing 6 coins we tossed 100 coins. This time we would have 101 columns in our graph. Now let us also suppose that for both of these graphs we have the same area and the same base line. It follows that these columns will be much narrower than those in Figure 10.1. If we have an infinite number, the columns would be very narrow and this binomial distribution would then approximate the normal distribution. Actually this binomial distribution is always discontinuous, but as the number of cases gets larger, this binomial curve gets closer and closer to the normal curve, but it never becomes, like the latter, continuous.

Parameters of the Binomial Distribution. Before we can use this binomial distribution in any problems, we have to learn how to compute several parameters which describe the curve. Notice now that we are using the term parameter because we are no longer dealing with samples but with population values. In Table 10.3 are shown the results of tossing eight coins at once, or if we wish to deal in educational terms, the chance results of answering

TABLE 10.3. Worksheet for the Computation of Binomial Parameters

(1) X	(2) f_e	(3) fX	(4) x	(5) x^2	(6) fx^2
8	1	8	4	16	16
7	8	56	3	9	72
6	28	168	2	4	112
5	56	280	1	1	56
4	70	280	0	0	0
3	56	168	−1	1	56
2	28	56	−2	4	112
1	8	8	−3	9	72
0	1	0	−4	16	16
Σ's	256	1024			512

eight true-false test items. In column (1), labeled X, we have all the possible scores or outcomes. Since we had an n of eight to begin with, there are nine values in this column. In column (2) we have the expected frequencies. These were obtained by the use of either the binomial theorem or Pascal's triangle. The third column is obtained by multiplying the values in each of the first two columns row by row. Column (3) is summed. If we wish to obtain the mean, we must divide this sum by the number of cases

$$m = \frac{\Sigma fX}{N}$$

$$m = \frac{1024}{256}$$

$$m = 4$$

In column (4) we have the deviation of each of the X-values in column (1) from this mean of 4. In column (5), each of these is squared and in column (6) each of the values in column (5) has been multiplied by the frequency for its row. This last column is then summed. Next we compute the standard deviation in the following manner:

$$\sigma = \sqrt{\frac{\Sigma fx^2}{N}}$$

$$\sigma = \sqrt{\frac{512}{256}}$$

$$\sigma = \sqrt{2}$$

$$\sigma = 1.42$$

It should be noticed that we are using different symbols for both mean and standard deviation at this time. Also we no longer divide the sum of the squares by $(N - 1)$ because we are no longer correcting samples for bias, but are dealing with population values.

If we are dealing with an actual problem, it is not necessary to go through the work shown in Table 8.3. Both of these parameters can be obtained by the use of the following formulas:

$$m = np \tag{10.2}$$

where

m = the mean of the binomial distribution.
p = the probability of the event's occurring.
n = the number of objects involved, or the exponent of the binomial.

For these data we have

$$m = np$$
$$m = 8(\tfrac{1}{2})$$
$$m = 4$$

which was the value obtained previously. The formula for the standard deviation of the binomial distribution is as follows:

$$\sigma = \sqrt{npq} \tag{10.3}$$

where all symbols are as previously defined.

For our problem we find the standard deviation as follows:

$$\sigma = \sqrt{8(\tfrac{1}{2})(\tfrac{1}{2})}$$
$$\sigma = \sqrt{2}$$
$$\sigma = 1.42$$

which is again the same as the value previously computed. The above formulas are used when the binomial distribution is taken as an approximation of the normal distribution. This should *only* be done when np or nq (whichever is the smaller) is equal to or greater than 5.

Use of the Formulas of the Binomial Distribution. Suppose that we construct a true-false test consisting of 26 items. We ask the question "What is the probability of getting a score of 20 or higher by chance alone?" that is, by guessing. We must assume that the examinee knows nothing about the material on which he is being tested. First we compute the mean and standard deviation for this distribution. Since np is larger than 5, we can go ahead and use the formulas for the binomial distribution.

$$m = np$$
$$m = 26(\tfrac{1}{2})$$
$$m = 13$$
$$\sigma = \sqrt{npq}$$
$$\sigma = \sqrt{26(\tfrac{1}{2})(\tfrac{1}{2})}$$
$$\sigma = \sqrt{6.50}$$
$$\sigma = 2.55$$

In Figure 10.2 we have drawn a curve with this mean of 13. Note that we have placed this score of 20 off to the right. The question is exactly how much of the area of the normal curve is above this score of 20. We have worked problems like this before. The first thing that we do is to change this raw score of 20 to a standard or *z*-score as follows:

$$z = \frac{X - m}{\sigma}$$

$$z = \frac{20 - 13}{2.55}$$

This standard score formula is based upon a normal curve where the distribution is continuous. Since we are dealing with a discrete, or noncontinuous distribution, we make a correction for the lack of continuity at this point. This is done by using, instead of the score of 20 in the formula, the lower limit of 20 which is 19.5. When we make this correction our work is much more accurate.

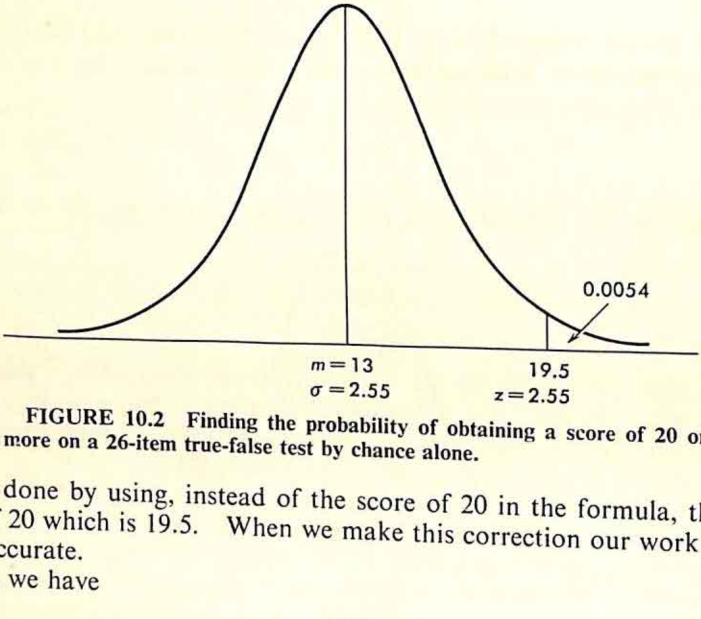


FIGURE 10.2 Finding the probability of obtaining a score of 20 or more on a 26-item true-false test by chance alone.

Then we have

$$z = \frac{19.5 - 13}{2.55}$$

$$z = \frac{6.5}{2.55}$$

$$z = 2.55$$

Going to the normal probability table (Table II in the Appendix) we find that .0054 of the area of the curve is above this standard score of 2.55. Then the chances are 54 in 10,000 that an individual can get 20 or more of these items correct by chance alone.

Next let us ask another question. This time we want to know what is the probability of obtaining a score of exactly 18 on this same true-false test by chance alone. This time refer to Figure 10.3. The lower limit of 18 is 17.5 and the upper limit is 18.5. We then want to find the area of the curve which falls between these two points. We first change both of these values to z-scores and then enter the normal probability tables to find the area cut off between the mean and each of these points. We then subtract the smaller of these areas from the larger. The remainder will be the area of the curve cut off by the two points.

$$z_1 = \frac{17.5 - 13}{2.55} \qquad z_2 = \frac{18.5 - 13}{2.55}$$

$$z_1 = \frac{4.5}{2.55} \qquad z_2 = \frac{5.5}{2.55}$$

$$z_1 = 1.77 \qquad z_2 = 2.16$$

Going to the normal probability table, we find the following areas between the mean and each of these points:

$$z_1 = 1.77 \quad .4616$$
$$z_2 = 2.16 \quad .4846$$

The difference between these two areas is .023. Then we can say that the chances are 23 in 1000 that an individual will obtain a score of exactly 18 on this 26-item true-false test.

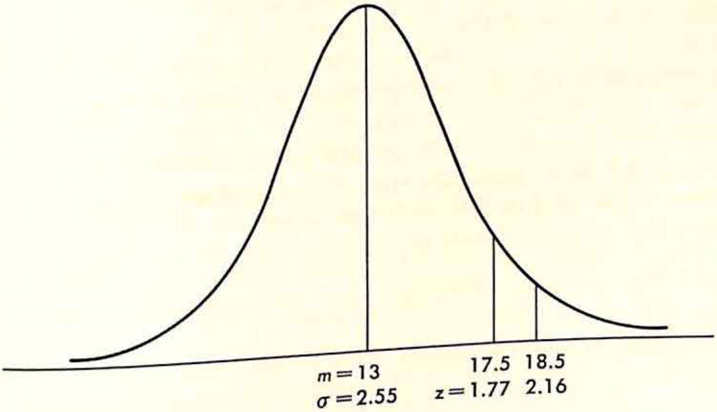


FIGURE 10.3 Finding the probability of obtaining a score of 18 on a 26-item true-false test by chance alone.

In psychological research, the binomial distribution is used to determine the ability of individuals to distinguish between different types of cola drinks, orange juices, cigarettes, oleo and butter, and the like. This is the way it is done. Suppose that we have four cola drinks and we set up the hypothesis that an individual is unable to separate cola A from the three others. Suppose that over a period of days we give an individual 60 trials and we observe that he identified cola A correctly 20 times. The question is whether this differs significantly from chance. We solve this problem exactly as we did the previous ones. First we compute the mean and standard deviations:[1]

$$m = np$$
$$m = 60(\tfrac{1}{4})$$
$$m = 15$$
$$\sigma = \sqrt{60(\tfrac{1}{4})(\tfrac{3}{4})}$$
$$\sigma = \sqrt{\frac{180}{16}}$$
$$\sigma = \sqrt{11.25}$$
$$\sigma = 3.35$$

Changing this observed frequency to a standard score and correcting for continuity we have

$$z = \frac{19.5 - 15}{3.35}$$
$$z = \frac{4.5}{3.35}$$
$$z = 1.34$$

Entering the normal probability table with a standard score of 1.34, we find that .0901 of the area of the normal curve falls to the right of an ordinate erected at this point. The chances then are 9 in 100 that this individual could identify cola A when samples of it are tasted along with three other kinds. We shall discuss the real significance of this probability value in a later chapter when we discuss the general problem of testing hypotheses. It might be noted here, however, that it is the usual practice to require a probability value of 5 in 100 or better before we are willing to accept the hypothesis that an individual is doing better than we would expect by chance.

Exercises

1. The following terms are discussed in this chapter; define each.
    (a) p
    (b) q
    (c) probability
    (d) mutually exclusive events
    (e) Pascal's triangle
    (f) discontinuous distribution
    (g) parameter
    (h) discrete data
    (i) binomial
    (j) independent events
2. What is the value of p for the occurrence of each of these events?
    (a) Drawing an ace in one draw from a well-shuffled deck of cards.
    (b) Drawing a heart in one draw from a well-shuffled deck of cards.
    (c) Winning the grand prize of a raffle in which 8000 tickets have been sold.
    (d) Obtaining a head from a tossed coin on the fourth toss when (1) the first three tosses turned up heads and (2) when the first three were tails.
    (e) Drawing a spade from a deck of cards on the fourth draw when the first three draws were spades and were not replaced in the deck.

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[1] Note that in this problem the probability of guessing the correct cola by chance is $1/4$ or .25.

3. What is the probability of these events?
   - (a) Obtaining five successive heads when tossing a coin.
   - (b) Throwing a twelve with a pair of dice.
   - (c) Throwing an odd number with a die.
4. How does the curve of the binomial distribution differ from the normal curve? How is it similar?
5. Smokers claim that they can identify the brand of cigarette which they customarily smoke. To test this an experiment is set up in which an individual is given four cigarettes to smoke, one of which is his favorite brand. All cigarettes are of the ordinary type. Of 110 subjects tested, 32 identify their own brand correctly. Is this a significantly larger number than would be expected by chance?
6. A 40-item multiple-choice test is made up of five-response items.
   - (a) If a person knows nothing about the material tested, what score could we expect by chance?
   - (b) What is the probability of getting a score of 12 by chance alone?
   - (c) What is the probability of getting a score 12 or higher by chance alone?
7. Bowles and Pronko gave 96 subjects a sample of each of three different cola drinks.[2] Coca Cola was correctly identified 39 times; Pepsi Cola, 36 times. Is either of these identifications significantly different from what would be expected by chance?
8. Suppose that you were going to take a 30-item true-false test on ancient Greek literature and further suppose that you know nothing about this subject. You decide to answer the items on the test by tossing a coin.
   - (a) What is the probability of your getting a score of 20 correct?
   - (b) What is the probability of getting a score of 20 or higher?
   - (c) What is the probability of getting all items correct?

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[2] *Journal of Applied Psychology*, 1948, **32**, 558–564.

11

SAMPLING AND MAKING INFERENCES

SAMPLING

A very important use of statistics is to make inferences about larger groups on the basis of information obtained upon smaller groups. In other words it may be stated that we wish to generalize about a population on the basis of what is known about one or more samples drawn from the population. The extent to which we can do this with any accuracy at all depends upon the adequacy of our sample or samples. We shall discuss first in this chapter samples and the characteristics of good ones.

In all the many different types of samples, the most important characteristic is *randomness*. By this we mean that every single sampling unit in the population has an equal chance of being drawn into the sample. A good example of simple random sampling took place when the National Selective Service Act was set up and individuals were arranged in order for their induction into various branches of the Armed Forces. Each individual, after registering with his local draft board, was given a number. These numbers ranged from one up to several thousand. Then a series of capsules was set up, each containing one of these numbers. These were then placed in a large goldfish bowl and the President of the United States, blindfolded, reached in and drew out a capsule. The number first drawn placed all individuals in each of the draft boards who held this number first in order for induction. Then a second number was drawn and the process continued until all those registered received an induction order number. Placing each number in a similar capsule, putting them all in the same bowl, mixing them, and then blindfolding the drawer made this type of sampling truly random.

Many times samples conducted on a national basis have not been random like this. The classical error is that which was made by the now defunct *Literary Digest* which was conducting a straw ballot to determine how the presidential election of 1936 was going to go. At that time, this national news magazine sent out ballots to a large number of individuals asking them whom they supported in the race for president, Roosevelt or Landon. On the basis of their returns, they stated that Landon was going to win easily over Roosevelt. In 1936 the Republicans carried two states, Maine and Vermont. Why was this poll far off? An investigation showed that the straw ballots were sent in general to two types of individuals, those who owned cars and those who had telephones. The names were taken from automobile registration lists and from telephone directories. Apparently the majority of individuals having autos and telephones in 1936 were Republicans.

A similar situation, though not so extreme, again appeared in 1948. As early as September of that year one of the national pollsters closed up shop saying that it was useless to do any more sampling as it was very obvious that Dewey was going to defeat Truman by a large majority. Again, it is history that the opposite happened. Along with poor sampling in some of the socioeconomic groups was an additional problem of a large number of voters who had not yet made up their minds as to how they were going to vote. In the past pollsters had relegated these undecided cases equally to both candidates. In 1948, a majority of these who were undecided apparently voted for Truman.

Such sampling as this is referred to as *biased* sampling. If I take a telephone book and select every twenty-fifth name as the individual to whom I am going to send a questionnaire and if I feel that I am studying the attitudes of any city on the basis of this questionnaire, I have a biased sample of the residents of this city. Any generalizations made would be valid about telephone subscribers but would be questionable about the city as a whole. This sample, though, could be considered as a random sample of telephone subscribers. Campus newspapers frequently run polls of how the students feel about various problems by asking the students a series of questions. The usual procedure is to go to a popular student hangout and ask the questions there. The results are reflections of the students who spend their time in such places and not the attitudes of the entire student body.

Suppose that we wanted to get a good picture of the attitudes of the entire student body on a certain campus issue. Suppose, to simplify things, that we deal with a campus of a large liberal arts college. The first thing to do would be to find out the number of freshmen, sophomores, juniors, and seniors in the entire student body. Then we would get the number of males and females in each class. If the issues involved aspects in which belonging to a fraternity or to a sorority was important, then we should know the number of students belonging in each of the classes and by sex in each class. Then we would turn all of these data into proportions or percentages. Our sample should be so drawn that it is made up of the same proportion of the different classes as is the entire student body. Also the proportion of males and females and fraternity and nonfraternity members in our sample should be the same as that of the entire student body. If instead of a liberal arts college, we had a large university, then we would have to see that the different schools of the university are represented proportionately in our sample.

After we decide the number of freshmen, male, fraternity men that we need in our sample, we should then have random sampling among all the freshmen, male, fraternity men. This would be true also of all the other groups too. Such a sample as this is described as being a *random stratified sample*. Sampling of this type is called for whenever a large heterogeneous group is being studied. If we wanted to study the attitudes of residents of a state, we would again begin by stratifying the population of the state. For example, we might begin by such a breakdown as rural and urban, breaking the latter into various groups on the basis of size of the population of each urban community.

There are other types of samples, too. Anyone who has been around a large university for any length of time usually becomes a part of a sample in some study or other. This is especially true of students in psychology classes, the elementary course being the most noted in this respect. Findings based upon such studies have to be interpreted in the most cautious and limited manner. Very few generalizations can be drawn as these university sophomores represent a biased sample of the larger population of youths of their own ages. Most of the results are limited to these groups of university students or to similar ones, if they exist in any other place.

There may be occasions where we study an entire group. It may be recalled that we have previously described a *population* or *universe* as an arbitrarily defined group. Suppose that we define our population as all of the seniors majoring in home economics in a university. Since this group is likely to be fairly small, we may use all of the individuals who make it up in the study. There is no sampling problem here. The numerical measures computed such as means and standard deviations would this time be described as parameters instead of statistics.

SAMPLING DISTRIBUTION OF THE MEAN

To begin our discussion of this suppose that we take a large population such as all 10-year-olds in the United States. With a population of this size and of this nature we can pull sample after sample and study them. Suppose that we take a group of 30 of these 10-year-olds in a certain school and give them an intelligence test. We find that the mean and standard deviation of this test are 98.7 and 17.8, respectively. Then we pull another sample of 30 cases in another school. This time we find that the mean and standard deviation of the intelligence test administered are 102.6 and 4.2. Suppose that we draw 998 more samples. From these we would have 998 more means and standard deviations. If we examine the means from these 1000 samples we see that we have quite a bit of variation. Suppose that we find the grand mean of these thousand means and that this comes to 100.1. We can use this grand mean as the best measure that we have of the population mean or parameter mean. Since it is impossible to measure all of these 10-year-olds, the true or population mean can never be known. The best we can do is to get estimates of it from samples. We now have the mean of 1000 means. Each of these sample means deviates from this grand mean by a certain amount. It is now possible to find the standard deviations of all of these sample means about the population mean. This standard deviation which is the standard deviation of a group of sample means about a population mean is now given a special name, *the standard error of the mean.*

All statistics have sampling distributions and thus all have standard errors. These standard errors give us an indication of the reliability of any given statistic. When the size of the standard error is small in relation to the units of measurement, it follows that our statistics will tend to vary less from sample to sample and thus we can have more confidence in our results when the size of the standard error for any sample is small.

In actual practice we never draw one sample after another until we have a large enough series to compute a standard error as we did above. Frequently, we have but one sample and draw all of our conclusions from this single sample. We get an estimate of the standard error of the mean by using the following formula:

$$s_{\bar{X}} = \frac{s}{\sqrt{N}} \qquad (11.1)$$

where

$s_{\bar{X}}$ = the standard error of the mean.
s = the standard deviation of the sample.
N = the number of cases in the sample.

Suppose that we have a sample with a standard deviation of 15.6 and based upon an N of 144. The standard error of the mean of this sample is as follows:

$$\begin{aligned} s_{\bar{X}} &= \frac{s}{\sqrt{N}} \\ &= \frac{15.6}{\sqrt{144}} \\ &= \frac{15.6}{12} \\ &= 1.3 \end{aligned}$$

An inspection of formula 11.1 reveals a very important characteristic of standard errors and that is that the larger the sample, the smaller the size of the standard error. This makes sense, for we would expect samples based upon larger samples to be more reliable than those based upon smaller ones. To be more exact we can say that the size of the standard error of the mean is inversely proportional to the square root of the number of cases in the sample and directly proportional to the standard deviation of the sample. This is what we see in formula 11.1. We can generalize from this and say that the size of the standard error of any statistic is inversely proportional to the number of cases in the sample upon which the statistic was computed.

The sample mean is usually described as being an unbiased estimate of the population mean. By this we mean that any given sample mean may be higher or lower than the population mean. If we take enough of these samples and average them out, the result will be an unbiased estimate of the population, that is, this result will tend to be neither too large nor too small when compared with the parameter mean. This is in contrast with the sample standard deviation which is described as a biased sample of the population standard deviation. If we draw a small sample, say, 20 cases, from a large population, these 20 cases are likely to come from near the center of the distribution rather than from the extremes because of chance. Hence the standard deviation of the sample will tend to be lower or smaller than that of the population. The smaller the sample, in reference to the size of the population, the greater this bias. We can then say that the sample standard deviation is a biased estimate of the population standard deviation because it tends to be smaller than the population value. It may be recalled that we corrected for this bias by using $N - 1$ in our formula for the sample standard deviation rather than just N as is found in some books.

Interpretation of the Standard Error of the Mean. Since a standard error is a standard deviation, we can interpret this standard error of the mean in the same manner as we would a standard deviation. Suppose that for this sample of 144 cases the mean on a certain test is found to be 70. We have calculated the standard error of this mean and found it to be 1.3. We do not know where the population mean for the trait or characteristic measured by this test lies, but we can say the chances are two out of three (or 68 in 100) that samples the same size as this one will not deviate from the population mean by more than 1.3 points. We are able to make this two-out-of-three probability statement because in the normal curve one standard deviation taken on each side of the mean cuts off a total of 68 percent of the area which would thus include 68 percent of the cases. To state this interpretation in another way, we can say that the chances are that two-thirds of sample means of this same size will not deviate more than 1.3 points from the true or population mean. If we wish to go further and measure off two standard errors of the mean on each side of the sample mean, we can then state that the chances are about 95 in 100 that means of samples of this same size would not deviate from the population mean by more than 2.6 units. In a similar manner, we could measure off three standard errors of the mean and then make a statement of almost certainty about the relation between the sample mean and the true mean, for it may be recalled that three standard deviations taken on each side of the mean include practically all of the area of the normal curve.

From this we can see that the smaller the size of the standard error of the mean, the more confidence we can have in our statistic. We say that our statistics are more reliable. By this we mean that our samples tend to be more similar from sample to sample. It should also be noted that the size of the standard error of the mean depends upon the size of the sample. This computed standard error of the mean of 1.3 only applies to a sample of 144 cases. If we had a smaller number in our sample, the size of the standard error would increase; if we had a larger sample, the value of the standard error of the mean would decrease.

SETTING UP CONFIDENCE LEVELS

Suppose that we begin by using the sample that we have previously discussed having a mean of 70. Since we do not know what the value of the population mean for this sample is, we can only make hypotheses about it. We can test these hypotheses until we get one that has an acceptable probability of being correct. Suppose that we start by assuming that the population mean is 74. This is shown in Figure 11.1. This sample mean of ours (70) deviates four units from this assumed population mean. We next can change this deviation of four to a standard score or to a z-score and see what the probability is that the population mean will be 74.

$$z = \frac{74 - 70}{1.3} = \frac{4}{1.3} = 3.08$$

A z-score of this magnitude when measured off from the mean cuts off in the smaller part of the curve .0010 of the area. Since a deviation of four could also take place in the other direction (that is, by assuming the population mean to be 66), we have to double this area or probability. We can say that the chances are 2 in 1000 that the population mean for this sample is as much as four units from the sample mean. So we can reject with confidence the statement that the population mean is 74.

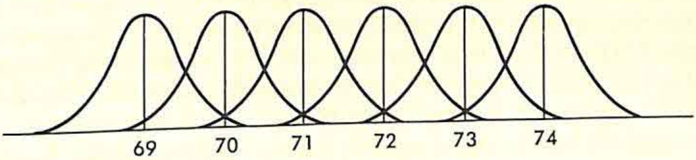

FIGURE 11.1 Sampling distributions related to different hypotheses made about the population mean when the sample mean is 70.

Suppose that next we hypothesize that the population mean is 73. This time we have a deviation of three, which when converted into a standard score results in a z-score of 2.31. Again referring to the normal probability table, we find that a standard score of this size when taken from the mean cuts off .0104 of the curve in the smaller area. Multiplying this by 2 to take care of a deviation of the same size on the other side of the mean results in a value of .0208. Again we can say that the probability that the population mean is as large as 73 or as small as 67 is about 2 in 100. Again, we reject with some confidence the hypothesis that the population mean is as large as 73.

Suppose that our next hypothesis is that the population mean for this sample is 72. This time our deviation is as large as two and this, when converted into a standard score, results in one of 1.54. The normal probability table reveals that this z-score of 1.54, when measured from the mean, includes .0618 of the area in the smaller portion. Doubling this results in a value of .1236. How we can say that the chances are about 12 in 100 that the population mean is 72. We cannot reject the hypothesis that the population mean is 72 with any degree of confidence.

Our last hypothesis about the population mean for this sample is that it is 71. Changing a deviation of this size to a standard score results in one of .77. When this is measured from the mean, the smaller portion contains .2206 of the area, and doubling this gives a value of .4412. Now the probability is about 44 in 100 that the population mean is 71 or 69. With a value like this, we cannot reject the hypothesis.

When we wish to make inferences about the population mean on the basis of a sample mean, we do not go through all the trial-and-error operations as we did above. It is the usual procedure, when testing an hypothesis, to use the 5 or 1 percent levels. By this we mean that we will reject these hypotheses about the mean when we can say that the probability is 5 in 100 or 1 in 100 that they are so. There is nothing sacred about the 5 or 1 percent level. When the probability value goes above 5, say 8, 10, or 20, most users of statistics feel that the odds are too much against them to use these values. At the 5 percent level, there are 5 chances in 100 that we are wrong, that is, that the hypothesis that we are rejecting should not be rejected. The 1 percent level cuts these chances of error to 1 in 100.

Since we usually operate at the 5 and 1 percent levels, we can set up limits for each of these levels in which we would expect the population mean to fall. These limits are referred to as *confidence limits.* Supoose that we set up the 5 percent confidence limits for our problem with the mean of 70 and the standard error of the mean equal to 1.3. If we go to the normal probability table we find that the standard score which cuts off 2.5 percent of the area in each tail of the normal curve is 1.96. It is important to remember that a z equal to 1.96 is the 5 percent level. Setting up the 5 percent confidence interval is done by taking a distance of 1.96 standard error units on each side of the sample mean as follows:

$$\begin{aligned} 5\% \text{ confidence interval} &= \bar{X} \pm (1.96)(s_{\bar{x}}) \\ &= 70 \pm (1.96)(1.3) \\ &= 70 \pm 2.55 \\ &= 67.45 - 72.55 \end{aligned} \tag{11.2}$$

We can now state that the chances are 95 in 100 for this sample that the population mean falls within the band 67.45–72.55.

The 1 percent confidence interval is set up in the same way. If we consult the normal probability table, we find that a z-score of 2.58 cuts off .005 percent of the area in the smaller portion. Doubling this to include the area in a deviation of the same size on both sides of the mean results in taking in 1 percent of the area of the curve. The procedure is the same as that above for the 5 percent interval except that this time 2.58 is used instead of 1.96.

$$\begin{aligned} 1\% \text{ confidence interval} &= \bar{X} \pm (2.58)(s_{\bar{x}}) \\ &= 70 \pm (2.58)(1.3) \\ &= 70 \pm 3.35 \\ &= 66.65 - 73.35 \end{aligned} \tag{11.3}$$

This time we can make the statement that the chances are 99 in 100 that the population mean for this sample falls within the band the limits of which are

66.65 and 73.35. It is logical that the confidence interval for the 1 percent value should be larger than that for the 5 percent value.

OTHER STANDARD ERRORS

The Standard Error of the Median. The standard error of the median is obtained by the following formula:

$$s_{\mathrm{Mdn}} = \frac{1.253s}{\sqrt{N}} \tag{11.4}$$

where all terms are as previously defined. If we compare this formula with that for the standard error of the mean we can see that the standard error of the median is about 25 percent larger than that of the mean. It may be recalled that it was stated in an earlier chapter that the mean was the most reliable measure of central tendency. By this we meant that it had the smallest standard error of the various measures of central tendency.

Standard Error of a Proportion. The standard error of a proportion is estimated by the following formula:

$$s_p = \sqrt{\frac{pq}{N}} \tag{11.5}$$

where $q = 1 - p$.

There are several limitations to the use of this formula. In the first place the sampling distribution of p is not the same for all values of p. It may be recalled that p can never be larger than 1, the values for this statistic falling between 0 and 1. It follows that if the population values of p are either large or small, that is, in either extreme of the distribution, it is impossible for the sampling distribution to be a normal one. This is shown in Figure 11.2 where the sampling distributions of population proportion values of .10, .50, and .95 are illustrated. The sampling distribution of both of the extreme proportions is skewed. The closer the value approaches .50, the more normal the curve of the sampling distribution will be.

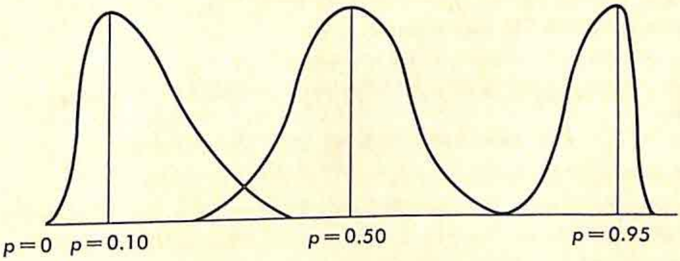

FIGURE 11.2 Sampling distributions of the various values of p.

Also this statistic should not be used with small samples. If Np or Nq, which ever is the smaller, is less than 10, this formula for the standard error of a proportion is not to be used. We shall see later that we have other statistics which we can use when our numbers are small and when the proportions of our samples tend to be in the extremes.

The Standard Error of a Percentage. Since a percentage is 100 times a proportion, we can write the formula for the standard error of a percentage as follows in terms of proportions:

$$s_P = 100\sqrt{\frac{pq}{N}} \tag{11.6}$$

or like this in terms of percents:

$$s_P = \sqrt{\frac{PQ}{N}} \tag{11.7}$$

where $Q = 100 - P$. This statistic has similar limitations to those of the standard error of a proportion.

Exercises

1. Which of the following selection techniques will result in random samples?
   (a) Population: all the residents of a given city.
   Sampling technique: stopping every third person who passes by a busy downtown street corner for one week.
   (b) Population: all the students in a large primary school.
   Sampling technique: selecting the first 75 students reporting to school on a Monday morning.
   (c) Population: all tomato plants in a field.
   Sampling technique: selecting one plant, blindfolded, from each square yard.
2. Suppose there are 1300 fifth grade students in a school system. You are given the task of estimating their arithmetic achievement.
   (a) Define precisely the population.
   (b) How would you sample this population?
   (c) Would your sampling cost less than the population measure?
   (d) It has been said that an investigator can learn more from a sample than from a universe. In a practical situation, does this make sense? Why?
3. What is a sampling distribution?
   (a) What is plotted on the vertical axis of a sampling distribution?
   (b) What is plotted on the horizontal axis?
4. What effect will sample size have on the size of the standard deviation of the sampling distribution of sample standard deviations. Is it possible to get the sample size so large that a spuriously large estimate is given?
5. On a *Watson-Glaser Critical Thinking Appraisal* 36 Ph.D. candidates in education achieve a mean score of 65. The standard deviation of the sample is 12.0.
   (a) What is the standard error of the mean?
   (b) What are the chances of drawing another similar sample with a mean of 69 or greater?
   (c) What are the 5 percent and 1 percent confidence intervals for the mean?
6. The scores of five chimpanzees on a finger dexterity test are as follows: 12, 14, 26, 9, 15.
   (a) What is the standard error of the median?
   (b) In this group 80 percent of the chimpanzees achieved a score higher than 10. What is the standard error of this percentage? Is this standard error meaningful?

12

TESTING DIFFERENCES BETWEEN MEANS

In this chapter and the next one we shall consider the various methods of testing the differences between two or more statistics. What we are basically interested in is whether or not the difference between two statistics is a real difference or whether it is merely a chance variation. We would expect the real difference to appear in future samples. These tests are referred to as tests of significance of differences.

DIFFERENCES BETWEEN MEANS—UNCORRELATED DATA

We shall start with one of the simpler situations in which we have two samples based upon a large N and in which there is no correlation between the two sets of data. Let us suppose that we have computed the following statistics for the two samples:

$$\begin{array}{ll} \bar{X}_1 = 58.6 & \bar{X}_2 = 56 \\ N_1 = 95 & N_2 = 100 \\ s_1 = 8.3 & s_2 = 6.2 \end{array}$$

The Null Hypothesis. We start a problem like this by stating what is known as a *null hypothesis*. Such a hypothesis is a statement of no difference. Actually we could state the hypothesis that we wish to test in various ways. We can state that there is no difference between these two means. Also we might say that these two means are the same. Basically we are saying in both of these null hypotheses that as far as the trait being measured here is concerned, these two sample means came from the same population. A basic assumption is that the samples are randomly drawn from a population in which the trait is normally distributed.[1] The purpose of stating a null

[1] When this assumption cannot be met, it is still possible to make tests of significance using some of the distribution-free statistics described in Chapter 18.

hypothesis is to have a hypothesis that can be tested. It may be noted that such logical procedures as these are most fundamental in experimental research in all areas of learning from agriculture to zoölogy.

We begin by stating a null hypothesis about these two means. We state that the difference between these two means is 0. By this we are saying that this difference between means, 58.6 − 56, or 2.6, is merely a chance deviation when the population value for the difference between these two means is 0. In other words, we now have set up a situation with a mean of 0 and a deviation of 2.6. Is this deviation too large to be accounted for by chance? By now the student should be familiar with the solution to this question by the use of the standard score z.[2] In this problem $\tilde{z}$ is the ratio of the difference between the means to the standard error of the difference between the means. For the standard error of the difference between the means we use the following for uncorrelated data:

$$s_{D_{\bar{x}}} = \sqrt{{s_{\bar{x}_1}}^2 + {s_{\bar{x}_1}}^2} \tag{12.1}$$

where

$s_{D_{\bar{x}}}$ = the standard error of the difference between two means for uncorrelated data.

$s_{\bar{x}_1}, s_{\bar{x}_2}$ = the standard errors of the two means the difference of which is being tested.

First, in solving this problem, we have to compute the standard error of each of the means. This is done as follows:

$$s_{\bar{x}_1} = \frac{s_1}{\sqrt{N_1}} \qquad\qquad s_{\bar{x}_2} = \frac{s_2}{\sqrt{N_2}}$$

$$s_{\bar{x}_1} = \frac{8.3}{\sqrt{95}} \qquad\qquad s_{\bar{x}_2} = \frac{6.2}{\sqrt{100}}$$

$$s_{\bar{x}_1} = \frac{8.3}{9.75} \qquad\qquad s_{\bar{x}_2} = \frac{6.2}{10}$$

$$s_{\bar{x}_1} = .85 \qquad\qquad s_{\bar{x}_2} = .62$$

Then using equation 12.1, we have

$$s_{D_{\bar{x}}} = \sqrt{{s_{\bar{x}_1}}^2 + {s_{\bar{x}_2}}^2}$$

$$s_{D_{\bar{x}}} = \sqrt{(.85)^2 + (.62)^2}$$

$$s_{D_{\bar{x}}} = \sqrt{.7225 + .3844}$$

$$s_{D_{\bar{x}}} = \sqrt{1.1069}$$

$$s_{D_{\bar{x}}} = 1.05$$

Next we change our deviation into standard score units as follows:

$$\tilde{z} = \frac{D_{\bar{x}}}{s_{D_{\bar{x}}}} \tag{12.2}$$

---

[2] We shall use z with a tilde over it ($\tilde{z}$) to denote that we are dealing now with sampling distributions.

where

D_x = the difference between the two means.

$s_{D_{\bar{x}}}$ = the standard error of the difference between the two means.

$$\tilde{z} = \frac{\bar{X}_1 - \bar{X}_2}{s_{D_{\bar{x}}}}.$$

$$\tilde{z} = \frac{58.6 - 56}{1.05}$$

$$\tilde{z} = \frac{2.6}{1.05}$$

$$\tilde{z} = 2.18$$

Evaluating the Results of the Test. There are two different approaches to handling a null hypothesis. In much experimental work, the research worker states ahead of time at what level of significance he will reject his null hypothesis. If he decides in advance that all decisions will be made at the 1 percent level, he merely evaluates his results at this level and disregards other results such as a difference being significant at the 5 or 10 percent level.

There are many other situations where we make a null hypothesis and merely observe the level at which the statistic is significant. In item analysis work we test how well an item separates good students from poor ones (that is, how well the item discriminates between two groups) by correlating the responses to each individual item with total test scores. Then we test each correlation coefficient to see whether or not it is a significant one. The best items have discrimination indices significant at the 1 percent level; then there are those significant at the 5 percent level. Sometimes to obtain enough items for a test we include other items which are significant at the 10 percent level.

To return to our particular problem, suppose that in advance we set our level of significance at the 5 percent level. Since the $\tilde{z}$ of 2.18 is between 1.96 and 2.58, we would then reject the null hypothesis at the 5 percent level. If we had set our level of significance in advance at the 1 percent level, the standard score of 2.18 would give us no reason for rejecting the null hypothesis.

If we were to use the observational approach, we would directly reject our hypothesis at the 5 percent level. It should be pointed out here that there is nothing sacred about the 1 and 5 percent levels. There may be situations in which an individual is willing to operate at the 10 percent level. Note, however, that when we reject the null hypothesis at the 1 percent level we have 1 chance in 100 of being wrong; at the 5 percent level, we have 5 chances in 100; and at the 10 percent level, we have 10 chances in 100 of being wrong. Perhaps the level of significance selected is related to how much of a gambler the experimental worker is. Note further that these standard scores of 1.96 and 2.58 are used repeatedly with tests of significance. The student will do well to associate them at once with their appropriate significance levels.

Two-Tailed Tests. We have just concluded a two-tailed test of significance. We started out by making a null hypothesis of no difference. We are merely trying to find out if there is a difference between the two means. Whether the first mean is larger or smaller than the second is of no concern. In using the two-tailed test, we use both ends of the curve as is shown in Figure 12.1. In this figure we see that a $\tilde{z}$-score of 1.96 has been marked off at each end of the curve. Each of these cuts off 2.5 percent of the total area of the curve. Doubling this gives us 5 percent, the value our computed $\tilde{z}$-score must equal for our results to be significant at the 5 percent level. In a similar fashion a standard score of 2.58 cuts off $^1/_2$ of 1 percent of the area on each side of the curve, making a total of 1 percent of the area. When we make a two-tailed test, a negative $\tilde{z}$-score is interpreted in the same manner as a positive one. When we reject a hypothesis using a two-tailed test, we say that we have rejected the hypothesis at the 1 or 5 percent *level.*

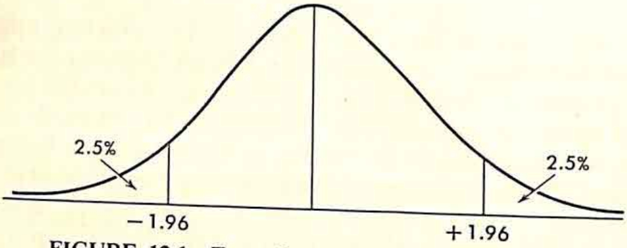

FIGURE 12.1 Two-tailed test at the 5 percent level.

One-Tailed Tests. There are times when on the basis of previous experiences with the variables in an experiment or on the basis of some cues, we are able to make what is called a one-tailed test. Suppose that we set up an experiment using a new reading technique. We have two groups: the control group, which is taught by one of the conventional reading methods, and the experimental group, which is taught by the new method. We have reasons to feel, on the basis of previous work with our method, that the experimental group should score higher on the reading test which they are given at the end of the semester. Our null hypothesis this time is that the experimental group has a mean achievement score which is equal to or less than that of the control group. If this hypothesis can be rejected, then it follows that we can make a probability statement about the alternative hypothesis, that the mean achievement score of the experimental group is higher than that of the control group. This time we are concerned only with one end of the curve as is shown in Figure 12.2. At the 5 percent level, we have 5 percent of the area all in one tail rather than having it equally divided into two tails. From the normal probability table we find that a z-score of 1.64 cuts off 5 percent of the area in the smaller part, and a standard score of 2.33 cuts off 1 percent of the area.

If we had reason to believe that the experimental group would score lower than the control group, we could set up the null hypothesis that the mean of the experimental group is equal to or greater than that of the control group. Rejecting this would leave us with the hypothesis that the experimental mean was less than that of the control mean. This time the area of the curve in which we are interested is the 5 or 1 percent of the area in the left-hand tail of the curve only.

When we were dealing with a two-tailed test, we noted that we rejected hypotheses at the 1 or 5 percent *levels*. To distinguish one-tailed tests from these, we say, when we reject a hypothesis using a one-tailed test, that we are rejecting at the 1 or 5 percent *points*.

Two Types of Error. When we reject a null hypothesis at the 5 percent level or point, there are 5 chances in 100 that we are wrong. This means that we are rejecting the hypothesis when it is actually true. In statistics this sort of error is referred to as a Type I error. We can reduce this type of error by using a more rigorous test. That is, we can decide to reject our null hypotheses at the 1 percent level only. Then there is only 1 chance in 100 of being wrong. If we so desire, we can further reduce our chances of error by going to the .1 percent level, and so on. But as we keep reducing our chances of making these so-called Type I errors, we are increasing our chances of making the Type II error. This type consists of not rejecting the null hypothesis when it should be rejected. There is a paradox here with no solution. As we decrease the possibility of making one type, we increase the possibility of making the other type. Most workers would rather be on the cautious side and tend to try to control Type I errors.

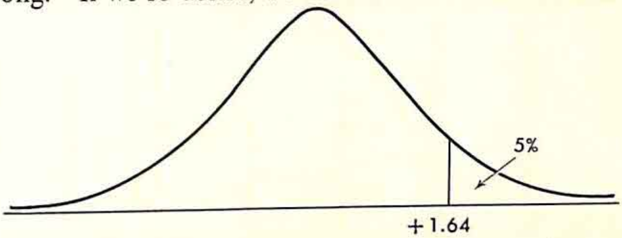

FIGURE 12.2 **One-tailed test at the 5 percent point.**

DIFFERENCES BETWEEN MEANS—CORRELATED DATA

We shall next consider the case in which the two sets of data are correlated. We say that data are correlated when they consist of two sets of measurements on the same individuals, repeated measurements of the same individuals with the same scale, or measures made upon twins or siblings. For example, suppose that we had two groups made up of brothers and sisters, the boys comprising one group and the girls the other. An intelligence test is administered to each group. These two sets of scores will be correlated. If data are correlated, such correlations have to be taken into account when tests of significance are being made.

In Table 12.1 are two sets of measurements (X and Y) on each of 20 individuals. At the bottom of the table are the two means and the rest of the table consists of the information necessary for the computation of the Pearson product-moment correlation coefficient by the use of deviations from the means. Substituting into the equation for r,

$$r = \frac{\Sigma xy}{\sqrt{(\Sigma x^2)(\Sigma y^2)}}$$

$$r = \frac{276}{\sqrt{(274)(442)}}$$

$$r = \frac{276}{\sqrt{121108}} = \frac{276}{348}$$

$$r = .79$$

TABLE 12.1. Scores on Two Variables for Twenty Individuals

X	Y	x	y	xy	x^2	y^2
18	20	1	1	1	1	1
16	22	−1	3	−3	1	9
18	24	1	5	5	1	25
12	10	−5	−9	45	25	81
20	25	3	6	18	9	36
17	19	0	0	0	0	0
18	20	1	1	1	1	1
20	21	3	2	6	9	4
22	23	5	4	20	25	16
20	20	3	1	3	9	1
10	10	−7	−9	63	49	81
8	12	−9	−7	63	81	49
20	22	3	3	9	9	9
12	14	−5	−5	25	25	25
16	12	−1	−7	7	1	49
16	20	−1	1	−1	1	1
18	22	1	3	3	1	9
20	24	3	5	15	9	25
18	23	1	4	4	1	16
21	17	4	−2	−8	16	4
Σ's = 340 $\bar{X}$ = 17	380 19	0	0	276	274	442

The procedure with this problem is basically the same as that for uncorrelated data. We again set up a null hypothesis of no difference between the two means, treat the difference between the two means as a deviation, and proceed to test whether or not a deviation as large as we have is or is not significant. In this problem we use the following formula for the standard error of difference between the two means:

$$s_{D_{\bar{x}}} = \sqrt{{s_{\bar{x}_1}}^2 + {s_{\bar{x}_2}}^2 - 2(r)(s_{\bar{x}_1})(s_{\bar{x}_2})} \qquad (12.3)$$

The student should be aware that this formula is the same as formula 12.1, the standard error of the difference of the means for uncorrelated data. When data are uncorrelated, r is 0, and the large third term under the radical sign disappears. The above formula may be considered the basic formula for the standard error of the difference between the means.

Next we shall compute the standard error of the difference between the means for our problem.

The first thing to do is to compute the standard deviation for each distribution and then to use each of these in calculating the standard error of each mean.

$$s_x = \sqrt{\frac{\Sigma x^2}{N - 1}} \qquad s_y = \sqrt{\frac{\Sigma y^2}{N - 1}}$$

$$s_x = \sqrt{\frac{274}{19}} \qquad s_y = \sqrt{\frac{442}{19}}$$

$$s_x = \sqrt{14.42} \qquad s_y = \sqrt{23.26}$$

$$s_x = 3.8 \qquad s_y = 4.8$$

Then the standard error of each mean follows:

$$s_{\bar{x}} = \frac{s_x}{\sqrt{N}} \qquad s_{\bar{y}} = \frac{s_y}{\sqrt{N}}$$

$$s_{\bar{x}} = \frac{3.8}{\sqrt{20}} \qquad s_{\bar{y}} = \frac{4.8}{\sqrt{20}}$$

$$s_{\bar{x}} = \frac{3.8}{4.47} \qquad s_{\bar{y}} = \frac{4.8}{4.47}$$

$$s_{\bar{x}} = .85 \qquad s_{\bar{y}} = 1.07$$

Substituting into the formula for the standard error of the difference between the means, we have

$$s_{D_{\bar{x}}} = \sqrt{(.85)^2 + (1.07)^2 - (2)(.79)(.85)(1.07)}$$

$$s_{D_{\bar{x}}} = \sqrt{.7225 + 1.1449 - 1.4370}$$

$$s_{D_{\bar{x}}} = \sqrt{1.8674 - 1.4370}$$

$$s_{D_{\bar{x}}} = \sqrt{.4304}$$

$$s_{D_{\bar{x}}} = .656$$

The last step is to compute the usual $\tilde{z}$-score as follows:

$$\tilde{z} = \frac{\bar{X} - \bar{Y}}{s_{D_x}}$$

$$\tilde{z} = \frac{17 - 19}{.656}$$

$$\tilde{z} = \frac{-2}{.656}$$

$$\tilde{z} = -3.05$$

A value of this magnitude with a two-tailed test allows us to reject our null hypothesis of no difference between the means at the 1 percent level.

Suppose that we did not wish to calculate the correlation coefficient but wished to determine the size of the standard error of the difference between the means directly and thus shorten the work. In Table 12.2 the same set of scores that appeared in Table 12.1 are again reproduced. By this method we go through the following steps:

1. Set up column (3) which is the difference between column (1) and column (2). It makes no difference which way these are subtracted except that the direction started must be continued throughout the entire process.

TABLE 12.2. Testing the Significance of the Difference Between Means When Data Are Correlated

(1) X	(2) Y	(3) D	(4) D^2
18	20	2	4
16	22	6	36
18	24	6	36
12	10	−2	4
20	25	5	25
17	19	2	4
18	20	2	4
20	21	1	1
22	23	1	1
20	20	0	0
10	10	0	0
8	12	4	16
20	22	2	4
12	14	2	4
16	12	−4	16
16	20	4	16
18	22	4	16
20	24	4	16
18	23	5	25
21	17	−4	16
		$\Sigma = 40$	$\Sigma = 244$

2. Sum this column of differences. Add the negative values and then subtract this sum from the sum of the positive values. Divide this sum by the number of pairs to compute the mean difference. For these data the mean difference is equal to 40 ÷ 20 which is 2. It should be noted that this mean difference is identical to the difference between the means which we have previously computed.
3. We are now going to take this mean difference, compute its standard error, and make the usual $\tilde{z}$-test. We first need, however, the sum of the squares for d and then the standard deviation for d.
4. Square all of the values in column (3). Then sum these values in column (4).
5. Compute the sum of the squares for d.

$$\Sigma d^2 = \Sigma D^2 - \frac{(\Sigma D)^2}{N}$$

$$\Sigma d^2 = 244 - \frac{(40)^2}{20}$$

$$\Sigma d^2 = 244 - 80$$

$$\Sigma d^2 = 164$$

6. Find the standard deviation of these differences.

$$s_d = \sqrt{\frac{\Sigma d^2}{N - 1}}$$

$$s_d = \sqrt{\frac{164}{19}}$$
$$s_d = \sqrt{8.6316}$$
$$s_d = 2.94$$

7. Then the standard error of the mean difference is found.

$$s_{\bar{d}} = \frac{s_d}{\sqrt{N}}$$
$$s_{\bar{d}} = \frac{2.94}{\sqrt{20}}$$
$$s_{\bar{d}} = \frac{2.94}{4.47}$$
$$s_{\bar{d}} = .658$$

8. Then the usual $\tilde{z}$-score is computed.

$$\tilde{z} = \frac{\text{mean difference}}{\text{standard error of mean difference}}$$
$$\tilde{z} = \frac{2}{.658}$$
$$\tilde{z} = 3.04$$

This value of 3.04 agrees within rounding error with that computed by the use of the correlation coefficient using formula 12.3.[3] There is no question but that this second method is easier and faster.

It is to the advantage of the research worker that he use the correct formula when testing differences between the means. Examination of the formula for the standard error of the difference between two means (12.3) shows that when the data are correlated the size of the standard error is reduced, depending upon the size of the correlation coefficient. When the standard error of difference decreases, the size of the z-score computed is larger, with the same value remaining in the numerator. Thus when the student uses the formula for uncorrelated data when he actually has correlated data, he is applying unnecessarily stringent tests to his data.

DIFFERENCES BETWEEN MEANS IN SMALL SAMPLES

In all of the work so far we have paid no attention to the size of the sample. However, in actual practice this is something that cannot be overlooked. When the sample size is large, we use the standard score, $\tilde{z}$, as the ratio of the difference between the means to the standard error of this difference and interpret this $\tilde{z}$-score by the use of the normal probability tables. When the size of the sample is small, especially when it is less than 25, a different technique must be used in interpreting the results.

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[3] If we had subtracted scores in column (2) from those in column (1), the mean difference would have been negative and hence our z-score would have been negative.

The t-ratio or Student's t. When the number of cases is small, we use the t-ratio or Student's t in interpreting our ratios instead of the normal probability tables. These ratios were published by W. S. Gosset in papers which were signed only "Student." An inspection of the t-table in the Appendix (Table III) shows that when $N(df)$[4] is large, the size of the 1 percent and 5 percent values of the t are the same as that for $\tilde{z}$. By the time that $N(df)$ has decreased to 100, the 1 percent value is 2.626 and the 5 percent is 1.984. From this point on down there is considerable difference between the values of t and those of $\tilde{z}$.

The t-ratio is defined in the same fashion as $\tilde{z}$. In other words it is a deviation divided by a standard deviation, the difference between the means being the deviation and the standard error of the difference between the means being the standard deviation. It follows then that it is not the computation of our work that has to change when we use small samples but the interpretation of our results. This is so because this t-statistic is not normally distributed when N is small. As the number of cases decreases, the sampling distribution of t becomes more and more peaked (leptokurtic) and the extremes of the tails become lifted from the base line, allowing for more cases in the tails. Student's t takes the shape of this distribution into account.

Degrees of Freedom. Notice that Table III in the Appendix is entered not with the size of the sample but with the number of degrees of freedom (*df*). Degrees of freedom means freedom to vary. Suppose we have six scores and the mean of these six scores is to be 10. This sixth score makes adjustments in the variation brought about by the first five scores and assures that the mean of the scores will be 10. For example, suppose we have five scores 10, 12, 18, 16, and 4. In order for the mean to be equal to 10 a sixth score must be 0. In another series, 2, 8, 4, 6, and 10, the sixth score must this time be 30 if the mean of the six scores is to be 10. In each of these cases we have 5 degrees of freedom. Five of the scores in the series may have any value, but the size of the sixth is determined, because the mean in each case is 10.

Suppose we take a case of 44 scores. First we compute the mean and then we take the deviation of each score from the mean and compute the standard deviation of the sample. In computing this mean we have used up 1 degree of freedom. We had 44 degrees of freedom to begin with, but now after computing the mean we have $(N - 1)$ or 43 degrees of freedom.

When data are paired as they are in Table 12.2, the number of degrees of freedom is equal to one less than the number of pairs. In Table 12.2 we then have 19 *df*. Going to the t-table we find that the 5 and 1 percent values are 2.093 and 2.861, respectively. For the data in Table 12.2 we obtained a $\tilde{z}$ of 2.97. Because of the size of N we really should have been working with t's here, but the small sample was used to make the illustration compact enough to fit into one table.

Setting Up Confidence Intervals for Small Samples. We have previously discussed the method for setting up confidence intervals for large sample.

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[4] Explained below.

To do this for the 5 percent level the confidence interval of the mean was obtained as follows:

$$(\bar{X} \pm (1.96)(s_{\bar{x}})$$

Suppose now that we have the following statistics for a set of data: $\bar{X} = 40$, $s = 6$, $N = 25$. We wish to set up the 5 percent confidence interval for this mean. We would compute the standard error of this mean which is $6/\sqrt{25}$ which equals 1.20. Instead of using 1.96 for the 5 percent value of $\bar{z}$, we go into the t-table with the appropriate number of degrees of freedom, 24 in this case, and obtain the t-value. We find this to be 2.49. So the 5 percent confidence interval for these data becomes

$$40 \pm (2.49)(1.20)$$
$$40 \pm 2.99$$
$$37.01 - 42.99$$

In the same manner the 1 percent confidence interval is set up using a t-ratio of 2.80 which is the 1 percent value for $df = 24$. These confidence intervals are interpreted in the same manner as are those for large samples.

Small Sample Techniques for Testing Differences Between Means. When the data are correlated, the usual technique is to set up the data as shown in Table 12.2, make the t-test, and then interpret the results using $(N - 1)$ degrees of freedom. When the data are uncorrelated, we proceed as follows. In Table 12.3 are shown the scores of two groups of ten individuals on a short test. The third and fourth columns are the squares of the first two columns. The means and the sums of the squares are shown at the bottom of the table. The sum of the squares for each distribution was obtained by the use of this formula:

TABLE 12.3. Scores of Two Groups of Individuals on the Same Test

X_1	X_2	X_1^2	X_2^2
16	4	256	16
12	8	144	64
22	10	484	100
16	14	256	196
14	12	196	144
10	4	100	16
20	10	400	100
18	12	324	144
10	4	144	16
22	12	484	144
Σ's = 160	90	2744	940
$\bar{X}$ = 16	9		
Σx^2 = 184	130		

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{N}$$

It is necessary that these sums be computed because they are used later in the computation.

We start with the usual null hypothesis of no difference between the means. This time we are going to use the t-test and this is defined as the ratio of the difference between the means divided by the standard error of the difference, to wit:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s_{D_{\bar{x}}}}$$

Instead of using the formula for the standard error of the difference between the means that we used previously, we use a modification of this which is said to be based upon the pooled variances of the two samples. Instead of using two separate variances from each of the samples as an independent estimate of the population variance, the two sample variances are pooled to give a better estimate of the population variance.

The F-Test. Before we proceed to pool two variances, we must make an F-test. F is defined as follows:

$$F = \frac{s_1^2}{s_2^2} \qquad (12.4)$$

where

s_1^2 = the larger of the two sample variances.
s_2^2 = the smaller of the two sample variances.

At this point we are testing a null hypothesis of no differences between the variances. If we can let the null hypothesis stand because of a nonsignificant F, then we can go ahead and pool the variance. If the F is found to be significant, formula 12.1 may be used in computing the standard error of the difference between the two means.

For this problem we now have to calculate the two variances. This follows:

$$s_{x_1}^2 = \frac{184}{9} \qquad s_{x_2}^2 = \frac{130}{9}$$

$$s_{x_1}^2 = 20.4 \qquad s_{x_2}^2 = 14.4$$

Then

$$F = \frac{20.4}{14.4} = 1.52$$

Entering the F-table, Table V of the Appendix, with 9 *df* for the larger variance (greater mean square) and 9 *df* for the smaller variance (lesser mean square), we find that our F-value of 1.52 is far below the 5 and 1 percent table values. Hence we accept the null hypothesis of no difference between the two variances and pool the variances.

The formula for the standard error of the difference between two means using the pooled variances is as follows:

$$s_{D_{\bar{x}}} = \sqrt{\frac{\Sigma x_1^2 + \Sigma x_2^2}{(N_1 + N_2) - 2}\left(\frac{1}{N^1} + \frac{1}{N^2}\right)} \qquad (12.5)$$

$$s_{D_{\bar{x}}} = \sqrt{\frac{184 + 130}{(10 + 10) - 2}\left(\frac{1}{10} + \frac{1}{10}\right)}$$

$$s_{D_{\bar{x}}} = \sqrt{\frac{314}{18}(.2)}$$

$$s_{D_{\bar{x}}} = \sqrt{3.48}$$

$$s_{D_{\bar{x}}} = 1.87$$

Then we make the t-test.

$$t = \frac{\bar{X}_1 - \bar{X}_2}{s_{D_{\bar{x}}}}$$

$$t = \frac{16 - 9}{1.87}$$

$$t = \frac{7}{1.87}$$

$$t = 3.74$$

For a problem like this, the number of degrees of freedom is $(N_1 + N_2) - 2$. In this case $df = 20 - 2 = 18$. Going to the t-table we find that for 18 df the 1 percent value is 2.878. Hence we can reject our null hypothesis of no difference between the two means with great confidence.

Comparison of Changes. There is another type of situation in which differences between means are tested that requires special treatment. Commonly in experimental work we set up two groups, an experimental one and a control group. Some sort of a pre-test or initial test is administered to both groups at the beginning of the experiment. Then another variable is introduced and applied to the experimental group but not to the control group. This might be the introduction of a new type of learning, a new drug, a new fertilizer for plants, a new food for pigs, or any other similar experimental variable. Then both groups are again given a similar test. This may be the same as the pre-test or one similar to it. This second test is referred to as the end-test.

We now have two means for both the control and the experimental groups, and on the basis of what we have learned in this chapter we might test these means the following ways:

1. We could take the two pre-tests and test these two means.
2. We could take the experimental group and compare the mean on the end-test with the mean on the pre-test.
3. We could take the control group and compare the mean on the end-test with the mean on the pre-test.
4. We could take the end-test for both groups and test these to see if the means are significantly different.

Suppose that we examine each of the above possibilities, one at a time. If we find that there are no differences between the two groups on the pre-tests, we might conclude that the two groups were equally matched, and thus our experiment had a good beginning. If we compare the mean on the end-test with that of the pre-test for the experimental group and find that there is a significant difference between the two, we might conclude that the application of the experimental variable brought about a change, especially if we found no significant differences between the means of pre- and end-tests of the control group. Again, if we find a significant difference between the end-tests of the two groups, we might conclude that the experiment worked, especially if there was no difference between the two groups on the pre-test. Any one of these conclusions that we might make is not necessarily true, because other factors may have brought about the change which occurred. At any rate, none of these methods gets down to an evaluation of the net change. This has to be done by testing the significance of the difference between the two changes or the changes in the two groups. In symbols this would be $D = D_E - D_C$

where

D_E = mean change in experimental group.
D_C = mean change in control group.

To do this we would compute our usual z-score which in this case would be equal to the difference between scores on the pre- and end-tests divided by the standard error of this difference. Thus:

$$\tilde{z} = \frac{D_E - D_C}{\sqrt{s_{D_E}^2 + s_{D_C}^2}} \tag{12.6}$$

Since we have demonstrated that the difference between the two means is equal to the mean difference, we can solve this problem by using the method of differences as shown in Table 14.2. If this method is used, it will always be correct as it is the method for correlated data. Frequently, there is a correlation between the two groups because the subjects entered into the two groups were carefully matched. Since it is to our advantage to use a formula for correlated data when a correlation exists, we must do so here.

Difference scores should be used with caution as they are apt to be more unreliable than the pre- and end-test scores from which they are derived. Frequently, much time and effort are spent in studying difference scores, the reliability of which is near zero.

OTHER TESTS OF DIFFERENCES

We have studied in detail some of the ways in which we test the differences between two means. Now we shall consider specifically differences between proportions and percentages. Before we do this, however, we should note that there is a general way of testing the differences between two statistics. We use a $\tilde{z}$- or a t-test, depending on the size of the sample. Both of these are defined as the difference between the two statistics that we are considering over the standard error of this difference. In the previous section it was the difference between the means divided by the standard error of the difference between the means. It could just as easily be the difference between the two medians or the two proportions. Of course, we have to use the correct formula for the standard error of the different statistics that we are using. Some of these formulas were presented at the end of Chapter 11.

Testing the Difference Between Two Proportions or Two Percentages. It may be recalled from Chapter 11 that the formula for the standard error of a proportion is

$$s_p = \sqrt{\frac{pq}{N}} \tag{12.7}$$

Then our test of significance would be the following:

$$\tilde{z} = \frac{p_1 - p_2}{\sqrt{{s_{p_1}}^2 + {s_{p_2}}^2}} \tag{12.8}$$

Before we go ahead and solve this we can make the solution easier if we change the denominator as follows:

$$s_{D_p} = \sqrt{{s_{p_1}}^2 + {s_{p_2}}^2} \tag{12.9}$$

$$s_{D_p} = \sqrt{\left(\sqrt{\frac{p_1q_1}{N_1}}\right)^2 + \left(\sqrt{\frac{p_2q_2}{N_2}}\right)^2}$$

Substituting formula 12.7 for its equivalent, we have

$$s_{D_p} = \sqrt{\frac{p_1q_1}{N_1} + \frac{p_2q_2}{N_2}} \tag{12.10}$$

This latter equation should be used in preference to the denominator of (12.8) because we can go into it directly with our p-values, and the computation of he two separate standard errors of the proportions is not necessary.

As an illustration suppose that we have the following data:

$$p_1 = .70 \qquad p_2 = .80$$
$$N_1 = 200 \qquad N_2 = 150 \quad r_{12} = .00$$

We first compute the standard error of the difference between the two proportions as follows:

$$s_{D_p} = \sqrt{\frac{(.70)(.30)}{200} + \frac{(.80)(.20)}{150}}$$

$$s_{D_p} = \sqrt{\frac{.21}{200} + \frac{.16}{150}}$$

$$s_{D_p} = \sqrt{.00105 + .00107}$$

$$s_{D_p} = .046$$

Then, making our test of significance, we have

$$\tilde{z} = \frac{p_1 - p_2}{s_{D_p}}$$

$$\tilde{z} = \frac{.70 - .80}{.046}$$

$$\tilde{z} = \frac{-.10}{.046}$$

$$\tilde{z} = -2.17$$

Since this value is larger than the 5 percent level (1.96), we can reject the null hypothesis of no difference between the two proportions at the 5 percent level.

When Samples Are Small or Proportions Are Extreme. The above technique works adequately when the N's are above 100. However, if either of the N's is less than 100 and either or both of the p's are extreme, that is, less than .10 or greater than .90, it is better to base the standard error of the difference on the proportion in the two groups combined. Since we are setting up a null hypothesis that the sample proportions are similar, we are justified in doing this.

Here are some more data.

$$p_1 = .90 \qquad p_2 = .80$$
$$N_1 = 40 \qquad N_2 = 30 \quad r_{12} = .00$$

First we compute p which is the proportion in the two groups combined. Thus p_1 is made up of 36 individuals (.90 × 40) and p_2 is made up of 24 individuals (.80 × 30). There is a total of 70 individuals (40 + 30). Then p is equal to $\frac{36 + 24}{70} = \frac{60}{70} = .86$.

Then

$$s_{D_p} = \sqrt{\frac{pq}{N_1} + \frac{pq}{N_2}} \tag{12.11}$$

or

$$s_{D_p} = \sqrt{pq\left(\frac{1}{N_1} + \frac{1}{N_2}\right)} \tag{12.12}$$

Substituting for the given data,

$$s_{D_p} = \sqrt{(.86)(.14)\left(\frac{1}{40} + \frac{1}{30}\right)}$$
$$s_{D_p} = \sqrt{(.1204)(.025 + .033)}$$
$$s_{D_p} = \sqrt{(.1204)(.058)}$$
$$s_{D_p} = \sqrt{.0069832}$$
$$s_{D_p} = .084$$

Then

$$t = \frac{p_1 - p_2}{s_{D_p}}$$
$$t = \frac{.90 - .80}{.084}$$
$$t = \frac{.10}{.084}$$
$$t = 1.19$$

This is not significant at the 5 percent level and hence our null hypothesis of no difference between the two proportions stands.

The above technique can be used with large samples as well as with small ones. With large samples the difference in the results will be trivial; with smaller samples, more reliable results are obtained using the last method.

With data like these, the number of degrees of freedom is equal to the sum of the number of degrees of freedom in each of the samples. For example, in the last problem, $df = (N_1 - 1) + (N_2 - 1)$ or $39 + 29$ which equals 68.

The student should note that the above solutions were for uncorrelated data, $r = .00$. When the data are correlated, the formula for the standard error of the difference between two proportions becomes

$$s_{D_p} = \sqrt{s_{p_1}{}^2 + s_{p_2}{}^2 - 2\,(r_{12})\,(s_{p_1})\,(s_{p_2})} \qquad (12.13)$$

McNemar (**10**) has developed a technique for testing the difference between two percentages that does not necessitate the computation of the correlation coefficient between the two variables. The data have to be tabulated into a two-by-two table as shown in Table 12.4. Suppose that we are interested in the differences of the responses that the same group of individuals make to two different attitude scale items. The responses of each individual have to be taken separately and each one entered into the table. Suppose that the first individual responded "Yes" to the first and second items. A tally would then be placed in the cell labeled *b*. The next individual responded "No" to the first item and "Yes" to the second one. This tally would be placed in the *d* cell. In this fashion the responses to all of the items by all of the individuals are tallied. Notice that the tallies placed in the *b* and *c* responses are those individuals who responded in the same direction to the two items. Cells *a* and *d* represent the individuals who answered the two items differently. In Table 12.4 are shown the responses of 120 individuals to two such items tallied in the manner just described.

TABLE 12.4. Responses of 120 Individuals to Two Attitude Scale Items

		Item 2		
		No	Yes	
Item 1	Yes	*a* 33	*b* 47	80
	No	*c* 25	*d* 15	40
		58	62	120

The test of significance is made in the following manner:

$$\tilde{z} = \sqrt{\frac{(a - d)^2}{a + d}} \qquad (12.14)$$

$$\tilde{z} = \sqrt{\frac{(33 - 15)^2}{33 + 15}}$$

$$\tilde{z} = \sqrt{\frac{(18)^2}{48}} = \sqrt{\frac{324}{48}}$$

$$\tilde{z} = 2.60$$

which is significant at the 1 percent level.

As a concluding note, since proportions and percentages are similar, all techniques applied here to proportions can be used equally well with percentages.

Exercises

1. Two groups in two high schools were given the *Minnesota Paper Formboard* with the following results:

A	*B*
$\bar{X}_1 = 32$	$\bar{X}_2 = 36$
$s_1 = 6.2$	$s_2 = 7.4$
$N = 144$	$N = 81$

Is there a significant difference between the two means?

2. A group of students was given a test of addition. A state of "anxiety" was induced and the arithmetic test readministered. The results are:

Pre-test	*End-test*
$\bar{X} = 70$	$\bar{X} = 69$
$s = 6$	$s = 5.8$
$N = 64$	$N = 64$

r between pre-test and end-test scores is .82.

(a) Is there a significant difference between the two sets of scores?

(b) Test the hypothesis that the mean on the end-test is significantly lower than the mean of the pre-test.

3. The following sets of scores were made by 16 individuals in a laboratory experiment on perception:

Individual	*Test 1*	*Test 2*
1	18	16
2	12	14
3	8	8
4	6	8
5	3	8
6	12	10
7	16	8
8	7	14
9	8	2
10	12	8
11	15	14
12	7	4
13	5	6
14	12	6
15	3	0
16	11	7

Test if there is a significant difference between the two means.

4. A group of 24 seniors in a technical high school achieve the following score on the *DAT Numerical Ability Test.*

25	17	29	29	26	24
27	33	23	14	21	26
20	27	26	32	20	32
17	23	20	30	26	12

A group of 25 seniors taking college preparatory work at another high school in the same city earn these scores on the same test.

21	26	28	31	14	27
29	23	18	25	32	23
16	21	17	20	26	23
7	18	29	32	24	17

(a) Using the appropriate test, determine if there is a significant difference between the two means.

(b) Suppose that you are only interested in knowing if the mean of the technical students is significantly higher than that of the college preparatory seniors. Use a one-tailed test to determine this.

5. The following are the responses of 100 students to two items on a test:

	Item 1	*Item 2*
Right	78	68
Wrong	22	32

Assuming that the responses to the two items are uncorrelated, test to see if there is a significant difference in the responses to the two items.

6. Thirty-two in a group of 40 and 18 in a group of 50 respond "Yes" to a certain attitude test item. Is there a difference in the responses of the two groups?

7. In 1958 there were in the U.S. Senate 11 members of the scholastic honor society, Phi Beta Kappa. Nine of these were Democrats and two were Republicans. There were 49 Democrats and 47 Republicans. Is the difference between the proportions of Phi Beta Kappans statistically significant?

8. (a) Why does one state a null hypothesis? Why not make a positive statement?

(b) Suppose that we state the null hypothesis: "The mean of the population from which Sample A is drawn is equal to the mean of the population represented by Sample B." We test the significance of the difference $\bar{X}_A - \bar{X}_B$. This difference is not significant at the 5 percent level. Therefore we cannot reject our null hypothesis. Can we accept it?

9. Set up the 1 and 5 percent confidence intervals for the means in Exercise 4. Interpret your results.

13

TESTING THE SIGNIFICANCE OF THE PEARSON r

The problem frequently arises after a correlation coefficient has been computed as to whether or not it is a measure of a real relationship or whether the coefficient at hand is merely a chance deviation from a population in which the r is 0. This is especially true when either the size of the Pearson r or the size of the sample, or both, is small.

Suppose that we compute the Pearson r for two sets of data based upon 82 cases and obtain an r of .30. We wish to know if this r indicates a real relationship between these two variables. We start with the usual null hypothesis that this obtained r does not differ from zero. By this we are saying that our obtained r is merely a chance deviation of the distribution in which the population r is 0. We test this r in the usual way by making a z- or t-test. Again these are defined as the ratio of a deviation to a standard deviation. The deviation in this case is our obtained r; the standard deviation is the standard error of this r. We obtain the standard error of the r by the use of the following formula:

$$s_{r_0} = \frac{1}{\sqrt{N-1}} \tag{13.1}$$

where

s_{r_0} = standard error when it is assumed to be 0.

N = number of pairs used in computing r.

$$s_{r_0} = \frac{1}{\sqrt{82-1}}.$$

$$s_{r_0} = \frac{1}{\sqrt{81}} = \frac{1}{9} = .11.$$

$$z = \frac{.30}{.11}.$$

$$z = 2.73$$

which with $N - 2$ degrees of freedom is significant at the 1 percent level.

Let us next take a case in which the N is small. Suppose that we have $N = 18$ and $r = .50$. The above technique is suitable when N is large, but when N is less than 30, the standard error of r has to be computed by the following formula:

$$s_{r_0} = \sqrt{\frac{1 - r^2}{N - 2}} \tag{13.2}$$

Then

$$s_{r_0} = \sqrt{\frac{1 - (.50)^2}{18 - 2}} = \sqrt{\frac{1 - .25}{16}} = \sqrt{\frac{.75}{16}}$$

$$s_{r_0} = \sqrt{.0469} = .216$$

We follow this with the t-test.

$$t = \frac{.50}{.216}$$

$$t = 2.31$$

This value with $(N - 2)$ df, or 16 df, is significant at the 5 percent level ($t_{05} = 2.12$ for $df = 16$).

A table has been developed which makes it unnecessary to go through either of the above procedures. This table (Table VI in the Appendix) is entered with $(N - 2)$ degrees of freedom, and the sizes of the r's needed to be significant for various degrees of freedom are shown. Suppose that we do the previous problem by the use of this table. N is 18; hence df is 16. For $df = 16$ we see that the r must be at least equal to .468 to be significant at the 5 percent level and .590 at the 1 percent level. Since the obtained r is .50 and falls between these two table values, we can reject the null hypothesis at the 5 percent level. In general the obtained r is compared with the table values. If it is larger than both, the null hypothesis of not being significant is rejected at the 1 percent level; if it falls in between the two table values, the null hypothesis is rejected at the 5 percent level; and if it is smaller than both table values, the null hypothesis stands.

The student should check the significance of any r that he is going to use in further computations. For example, in testing the difference between means, the r can rapidly be checked by the use of Table VI and the appropriate formula can be used in the calculation of the standard error of the difference between the two means.

TESTING THE DIFFERENCE BETWEEN TWO r's

The Sampling Distribution of r. We shall first consider the sampling distribution of r. It may be recalled that the size of r varies between -1 and $+1$. Since this is so, the sampling distribution of r cannot be normal. Suppose that we take the distribution of the sample r's when the population value is .90. Because of the location of this parameter on the continuum of r, it is impossible for the sampling distribution to be normal. No sample value can be larger than 1. But on the other side of .90 the size of the sample r's can extend downward with no limitation. Such a sampling distribution will be negatively skewed. This is true of all high correlation coefficients. Skewness for high negative r's will be positive. Actually the sampling distribution of r is roughly normal when the parameter size is .00. The further the population values are from this, the more skewed is their sampling distribution. This is shown in Figure 13.1. Since these sampling distributions are distorted, it is recommended that the usual z or t *not* be used in testing the difference between two r's.

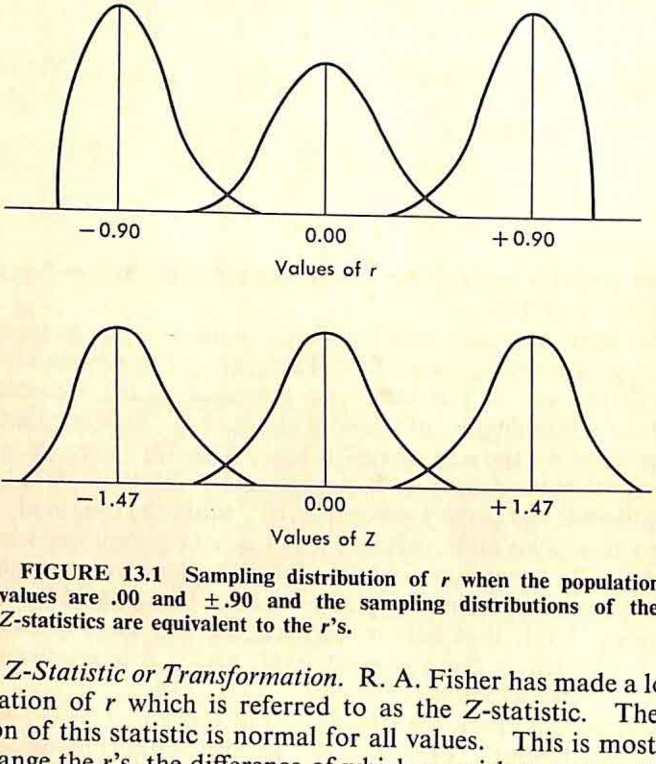

FIGURE 13.1 Sampling distribution of r when the population values are .00 and ±.90 and the sampling distributions of the Z-statistics are equivalent to the r's.

Fisher's Z-Statistic or Transformation. R. A. Fisher has made a logarithmic transformation of r which is referred to as the Z-statistic. The sampling distribution of this statistic is normal for all values. This is most useful for we can change the r's, the difference of which we wish to test, to Z's, test the difference between the two Z's, and, if this is significant, conclude that the difference between the two correlation coefficients is significant. The standard error of Fisher's Z-transformation is as follows:

$$s_Z = \frac{1}{\sqrt{N - 3}} \tag{13.3}$$

Testing the Difference Between Two Correlation Coefficients for Uncorrelated Data. Suppose that we have the following data:

$r_{12} = .85$ $\qquad$ $r_{34} = .78$

$N = 103$ $\qquad$ $N = 147$

We wish to test the null hypothesis of no difference between these two correlation coefficients. The first thing that we do is change each of these into Fisher's Z-statistic by the use of Table VII in the Appendix. This gives us

$$r_{12} \quad .85 \quad = \quad Z \quad 1.26$$
$$r_{34} \quad .78 \quad = \quad Z \quad 1.05$$

The problem now is one of testing the difference between these two Z's. The standard error of the difference between two Z's is written as shown in 13.4, but we simplify it to the form of equation 13.5 before we use it.

$$s_{D_Z} = \sqrt{s_{Z_1}{}^2 + s_{Z_2}{}^2} \tag{13.4}$$

$$s_{D_Z} = \sqrt{\left(\frac{1}{\sqrt{N_1 - 3}}\right)^2 + \left(\frac{1}{\sqrt{N_2 - 3}}\right)^2}$$

$$s_{D_Z} = \sqrt{\frac{1}{N_1 - 3} + \frac{1}{N_2 - 3}} \tag{13.5}$$

$$s_{D_Z} = \sqrt{\frac{1}{103 - 3} + \frac{1}{147 - 3}}$$

$$s_{D_Z} = \sqrt{\frac{1}{100} + \frac{1}{144}}$$

$$s_{D_Z} = \sqrt{.01 + .0069} = \sqrt{.0169}$$

$$s_{D_Z} = .13$$

Next we make the usual test of significance.

$$\tilde{z} = \frac{Z_1 - Z_2}{s_{D_Z}}$$

$$z = \frac{1.26 - 1.05}{.13}$$

$$z = \frac{.21}{.13} = 1.6$$

Since this is less than 1.96, it is not significant. Hence, since there is no significant difference between the two Z-statistics, we conclude that there is no significant difference between the two correlation coefficients.

If the situation exists in which variable 1 is correlated with variable 2, and variable 1 is also correlated with variable 3, and all measurements made upon the same sample, the above technique cannot be used, for here there has to be taken into account the relationship between the pairs of r's. Once again we are dealing with correlated data. McNemar (**10**) has suggested a t-test that can be made when this situation occurs.

$$t = \frac{(r_{12} - r_{13})\sqrt{(N - 3)(1 + r_{23})}}{\sqrt{2(1 - r_{12}{}^2 - r_{13}{}^2 - r_{23}{}^2 + 2r_{12}\,r_{13}\,r_{23})}} \tag{13.6}$$

This is interpreted by going into the t-table with $N - 3$ degrees of freedom.

Averaging Pearson r*'s.* Since Pearson r's are not equal units of measurement, they cannot be added and averaged. To find an average correlation coefficient, change each r to its respective Z using Table VII. Then average these Z's, and using the same table, convert this Z back to an r. This is the average r. For example,

	r	Z	
	.68	.83	
	.77	1.02	
	.86	1.29	
	.79	1.07	
	.92	1.59	
Sums	4.02	5.80	
Means	.80	1.16	which converts to an r or .82

The sum and mean in the left-hand column are merely inserted to show that there is a difference between the two techniques. If the r's were all fairly similar in size, little would be gained by using this technique. Also if they were all small, they could be averaged as is, for an inspection of Table VII reveals that up to an r of .25, the values of r and Z are similar.

Exercises

1. A student computes an r of .32 with 22 cases. By use of the appropriate formula test the significance of this r. Check your results by using the tables for testing the significance of an r.
2. $N = 103$. What is the standard error of a Fisher's Z which corresponds to an r of .90?
3. We find r' of $-.60$ and $-.51$, N being 85 in both sets of data. Using the formula for uncorrelated data, test the hypothesis of no difference between these two r's.
4. The following correlations are found among three variables in the same sample of 103 individuals:

	A	B	C
A	—	.76	.93
B	—	—	.85
C	—	—	—

Is the correlation between A and B significantly different from the correlation between A and C?

5. On five independent samples the following r's were obtained:

.93
.81
.63
.78
.68

What is the best estimate of the population r?

14

χ^2—CHI SQUARE

Chi square is a test of significance. This statistic is a very useful one in research because no particular assumptions have to be made about the shape of the distributions of the frequencies being tested. It is most commonly used when data are in frequencies such as in the number of responses in different categories. It can be used with any data that can be reduced to proportions and percentages. The social psychologist involved in research upon the attitudes of people finds this statistic a very useful one.

As a simple illustration suppose that we toss a fifty cent piece in the air 100 times and record our results. We observe that 40 heads and 60 tails appear. We refer to these frequencies as the observed frequencies. In a previous chapter we gave these the symbol f_o. However in this chapter we shall, for the sake of convenience, refer to these observed frequencies with the symbol O. Next we make the usual null hypothesis that this distribution of 40 heads and 60 tails does not differ from what we would expect by chance, that is, 50 heads and 50 tails. These frequencies are called the expected, theoretical, or hypothetical frequencies. In this chapter we shall refer to these using the symbol E.

We test our hypothesis by using the general formula for chi square.

$$\chi^2 = \sum\frac{(O - E)^2}{E} \tag{14.1}$$

$$\chi^2 = \frac{(40 - 50)^2}{50} + \frac{(60 - 50)^2}{50}$$

$$\chi^2 = \frac{(10)^2}{50} + \frac{(10)^2}{50}$$

$$\chi^2 = \frac{100}{50} + \frac{100}{50}$$

$$\chi^2 = 2 + 2$$

$$\chi^2 = 4.00$$

Formula 14.1 demands that we take each observed frequency, subtract from it the corresponding expected frequency, square the difference, and divide the result by the expected frequency. The sum of these is chi square.[1]

Table IV in the Appendix is used for the interpretation of chi square. We enter this table with $df = 1$ (this will be explained later) and compare our computed value of chi square with the table values. We note that the 5 percent value of chi square for 1 *df* is 3.841. Our computed value of 4.00 is larger than this but less than the 2 and 1 percent values; hence we reject our hypothesis of no difference at the 5 percent level of significance. We can be fairly confident that these results are different from those produced by chance alone.

Suppose that we take another example. This time we toss a die 144 times and we observe that a five-spot appears 36 times. We set our observations up as follows:

	O	*E*
five-spots	36	24
Not five-spots	108	120
Total	144	144

Since there were 36 five-spots, it follows that there were 108 observations that were not five-spots. Our expected frequencies are based upon chance. Since the probability of tossing a five-spot is $^1/_6$, we put down our expected frequencies in that proportion, which in this case gives us 24 and 120. It is important to note that in a chi square problem, the sum of the observed frequencies has to equal the sum of the expected frequencies. For these data

$$\chi^2 = \frac{(36 - 24)^2}{24} + \frac{(108 - 120)^2}{120}$$

$$\chi^2 = \frac{12^2}{24} + \frac{12^2}{120} = \frac{144}{24} + \frac{144}{120}$$

$$\chi^2 = 6.0 + 1.2$$

$$\chi^2 = 7.20$$

which with $df = 1$ is significant at the 1 percent level.

Chi Square in a 2 × 2 Table. One of the commonest uses of chi square is found in the so-called 2 × 2 table or fourfold table. In Table 14.1 are shown the "Yes-No" responses of two groups to an item on an attitude

TABLE 14.1. Chi Square in a 2 × 2 Table

	O			*E*		
	Yes	No		Yes	No	
Group 1	40 *a*	28 *b*	68 *k*	45	23	68
Group 2	50 *c*	18 *d*	68 *l*	45	23	68
	90 *m*	46 *n*	136	90	46	136

---

[1] Note that we consider both the "heads" and the "nonheads" or negative responses in solving for χ^2.

inventory. The observed frequencies have been entered in the left-hand table. Such a table as this is referred to as a *contingency table*. Assuming that chance alone is operating, the expected frequencies are set up, since the groups are equal in size, by assigning half of the "Yes" responses and half of the "No" responses to each of the two groups. It might be noted that it is not necessary to have the groups of equal size for the use of this technique. Chi square is computed for these data in the same manner as it was for the previous problem, except now we have four sets of cells to compare.

$$\chi^2 = \frac{(40 - 45)^2}{45} + \frac{(28 - 23)^2}{23} + \frac{(50 - 45)^2}{45} + \frac{(18 - 23)^2}{23}$$

$$\chi^2 = \frac{25}{45} + \frac{25}{23} + \frac{25}{45} + \frac{25}{23}$$

$$\chi^2 = .555 + 1.087 + .555 + 1.087$$

$$\chi^2 = 3.285$$

Degrees of Freedom. In a chi square problem degrees of freedom are determined generally by the use of the following formula:

$$df = (r - 1)(c - 1) \tag{14.2}$$

where

r = the numbers of rows in the contingency table.

c = the number of columns in the contingency table.

For the data in Table 14.2, this reduces to

$$df = (2 - 1)(2 - 1) = 1 \times 1 = 1$$

The student will note that as soon as one of the cell values is determined, the marginal values remaining the same, the frequencies of the other cells are then fixed. For this table then there is only 1 *df.* Obviously, this formula cannot be applied to the two problems that we worked first for then we would have 0 *df.* In a 2 × 1 table, *df* is always equal to 1. Entering the chi square table in the Appendix we find that for $df = 1$ a chi square of 3.285 is not significant.

Another method for finding the value of chi square in a 2 × 2 table is by the use of the following formula which avoids the computation of the expected frequencies.

$$\chi^2 = \frac{N[(ad) - (bc)]^2}{(k)(l)(m)(n)} \tag{14.3}$$

where the various letters are shown in the left-hand contingency table in Table 14.1 and N is the total number of frequencies. Solving the data in Table 14.1 by the use of this formula

$$= \frac{136[(40)(18) - (28)(50)]^2}{(68)(68)(90)(46)}$$

$$= \frac{136(720 - 1400)^2}{19143360} = \frac{62,886,400}{19,143,360}$$

$$= 3.285$$

which is identical to the answer computed by the other method.

Obtaining the Expected Frequencies. To illustrate how the expected frequencies are obtained, we shall use the material in Table 14.2. The cell frequencies of this 3 × 3 contingency table are designated by the letters within the cells. The rows and columns are summed and the total number of frequencies is represented by T. We are going to obtain these expected frequencies by manipulating the margins. To obtain the expected frequency for the cell in the extreme upper-left cell, we multiply M by R and divide this by the total number of cases T. The E for the cell below this is obtained by multiplying M by S and dividing by T. After we finish the left-hand column of the contingency table, we move to the right, and to obtain the E for the top middle cell we multiply N by R and divide this by T. In this manner all the expected cell frequencies in the contingency table may be determined. If the student wishes he may go back to Table 14.1 and, if the method just described is applied, he will compute the same expected frequencies as are noted there.

TABLE 14.2. Obtaining the Expected Frequencies

O

a	b	c	R
d	e	f	S
g	h	i	W
M	N	P	T

E

$\frac{MR}{T}$	$\frac{NR}{T}$	$\frac{PR}{T}$	R
$\frac{MS}{T}$	$\frac{NS}{T}$	$\frac{PS}{T}$	S
$\frac{MW}{T}$	$\frac{NW}{T}$	$\frac{PW}{T}$	W
M	N	P	T

Chi Square When Frequencies Are Small and $df = 1$. When any of the expected frequencies is small, say less than 10, the chi square computed is likely to be an overestimate. With $df = 1$, a correction called Yates' correction for continuity is applied. This correction is used because the distribution of chi square is discrete, whereas the values obtained by the formulas result in continuous distributions. When frequencies are large this difference is of no importance; but as the frequencies become small, Yates' correction should be applied. When we work the data by formula 14.1, the corrected formula becomes

$$\chi^2 = \sum \frac{(|O - E| - .5)^2}{E} \tag{14.4}$$

In the above formula what is actually happening is that each O which is larger than E is decreased by .5 and each O which is less than E is increased by .5.

When Yates' correction is applied to formula 14.3, this formula becomes

$$\chi^2 = \frac{N\left(|ad - bc| - \frac{N}{2}\right)^2}{(k)(l)(m)(n)} \tag{14.5}$$

The parallel lines around the $(O - E)$ and the $(ad - bc)$ indicate that it is the absolute values that are to be reduced by .5. If $(O - E)$ were -3.5, this would be reduced to -3.

To illustrate the use of Yates' correction, we shall work the data in Table 14.3. In Table 14.3 are shown the expected frequencies which result in

$$\begin{aligned}\chi^2 &= \left[\frac{(23 - 19 - .5)^2}{19} + \frac{(2 - 6 - .5)^2}{6}\right](2)^* \\ &= \left[\frac{(3.5)^2}{19} + \frac{(3.5)^2}{6}\right](2) \\ &= (.6447 + 2.0417)(2) \\ &= (2.6864)(2) \\ &= 5.3728\end{aligned}$$

TABLE 14.3. The Use of Yates' Correction in Chi Square

O

	Yes	No	
Group 1	23 *a*	2 *b*	25 *k*
Group 2	15 *c*	10 *d*	25 *l*
	38 *m*	12 *n*	50 *N*

E

Yes	No	
19	6	25
19	6	25
38	12	50

which, with $df = 1$, is significant at the 5 percent level. It might be noted here that the chi square when computed for these data, not using Yates' correction, is 7.017 which is significant at the 1 percent level.

Using formula 14.5, this problem is worked as follows:

$$\begin{aligned}\chi^2 &= \frac{50\left(\left|(23)(10) - (15)(2)\right| - \frac{50}{2}\right)^2}{(25)(25)(38)(12)} \\ &= \frac{50(200 - 25)^2}{285000} \\ &= \frac{1531250}{285000} \\ &= 5.37\end{aligned}$$

When E values are less than 5, and especially when they are around 2, even Yates' correction for continuity is not too good, and a method known as Fisher's exact method must be used. This will not be discussed here, but it may be found in McNemar (**10**, p. 241). This method is a complicated one. The student might find it to his advantage to increase the size of his N rather than be concerned with frequencies of 1 or 2.

---

* Since the other two cells would be identical to the first two, we multiply by 2.

Chi Square and z. When $df = 1$, $\chi^2 = z^2$. This may be illustrated by using the following data:

O	E
36	24
108	120
144	144

We have already found that chi square for these data is 7.20. Since the above data resulted from the toss of a die, the probability that any particular face of the die will appear is $^1/_6$. Applying the formulas for the binomial (10.2 and 10.3),

$$m = Np \qquad \sigma = \sqrt{Npq}$$

$$m = (144)\tfrac{1}{6} \qquad \sigma = \sqrt{(144)(\tfrac{1}{6})(\tfrac{5}{6})}$$

$$m = 24 \qquad \sigma = \sqrt{720/36} = \sqrt{20}$$

$$\sigma = 4.472$$

We next consider our observed frequency of 36 five-spots as a deviation from the mean of 24 and apply a z-test to see if this deviation is large enough to be a significant one.

$$z = \frac{36 - 24}{4.472} = \frac{12}{4.472}$$

$$z = 2.6833$$

$$z^2 = 7.20$$

which was the value previously computed for chi square. For certain problems, then it follows that we have a choice as to which statistic we use in making a test of significance. It must be remembered that this relationship between z and chi square holds only when $df = 1$.

Chi Square in a Table Larger Than 2 × 2. Suppose that we have the responses of three groups to a single item of an attitude inventory scale. The responses were recorded as "Strongly Agree," "Agree," "No Opinion," "Disagree," and "Strongly Disagree." The frequencies of the responses for the three groups are shown in Table 14.4. There is nothing different in the solution of the chi square for these data than that for the 2 × 2 table, other than the amount of work involved. First the marginal values for the observed frequencies are obtained. Then using these margins, the expected frequencies are determined. For this problem these were carried to the nearest tenth. The chi square component for each cell is then determined, these components are summed, the chi square table in the Appendix is entered with $(3 - 1)(5 - 1)$ or 8 degrees of freedom, and the appropriate conclusion is drawn.

The resulting chi square is large and we reject the null hypothesis of no difference among the three groups at the 1 percent level. We now have strong reason to believe that there is a difference among the three groups. But where? Further tests have to be made to see if each group differs from the other or if two of them are similar, both differing from a third. A chi

TABLE 14.4. Chi Square in a Large Table

O

	SA	A	NO	D	SD	
Group 1	12	18	4	8	12	54
Group 2	48	22	10	8	10	98
Group 3	10	4	12	10	12	48
	70	44	26	26	34	200

E

Group 1	18.9	11.9	7.0	7.0	9.2	54
Group 2	34.3	21.6	12.7	12.7	16.6	97.9
Group 3	16.8	10.5	6.3	6.3	8.2	48
	70	44	26	26	34	200

$$\chi^2=\frac{(12-18.9)^2}{18.9}+\frac{(48-34.3)^2}{34.3}+\ldots+\frac{(12-8.2)^2}{8.2}$$[a]

$= 2.519 + 5.472 + \ldots + 1.761$

$= 34.209$ which with $df = 8$ is significant for χ^2 with $df = 8$, at the 1% level $= 20.09$

[a] Twelve of the components are not shown.

square test can be now made, taking one group at a time versus another group. If the overall chi square had not been significant, we could have stopped at that point. But if we have evidence that differences are present, we should take additional steps to find out where they are.

Small Frequencies in Large Tables. Suppose, using the same terminology as used in Table 14.4, we have the following responses of two groups to an attitude test item:

	SA	A	NO	D	SD
Group 1	2	15	12	18	1
Group 2	1	14	8	13	2

Before we start we should combine the frequencies in the "Strongly Agree" and "Agree" categories and those also in the two disagree categories and apply chi square to a contingency table that looks like this:[2]

	A	NO	D
Group 1	17	12	19
Group 2	15	8	15

---

[2] A good rule to follow is to do this when any *E* is less than 5.

Another situation like this may arise:

	SA	A	NO	D	SD
Group 1	24	46	2	14	8
Group 2	16	13	3	34	16

For these data it would be best to discard the "No Opinion" category and run the chi square test using the other four columns.

The next situation contains the responses of students in five different schools of a university to a five-response attitude test item.

	SA	A	NO	D	SD
Liberal Arts	16	32	12	37	18
Engineering	23	15	7	23	5
Home Economics	6	14	6	13	6
Fine Arts	9	23	6	7	8
Education	2	3	4	10	1

In the above contingency table it will be noted that the number of respondents in Education is small. There are three possibilities in handling data like this. First the Education data can be dropped from the analysis. Secondly, the frequencies of the Education group can be combined with those of some other group if there is any logic for such a combination. In this situation Education could be combined with Liberal Arts, for on some campuses these two groups are together. Thirdly, and probably the best, more individuals in Education should be sampled. Yates' correction is not applied to these larger tables.

Chi Square Test for Correlated Proportions. In Chapter 12, we made a z-test (or a t-test) for the difference between two proportions when the data are correlated. The data used in Chapter 12 are reproduced below:

		Item 2	
		No	Yes
Item 1	Yes	*a* 33	*b* 47
	No	*c* 25	*d* 15

In this table we have the responses of 120 individuals to two test items. The data are arranged in the table by taking into account the agreement and disagreement of the responses of the individuals to the two items.

The formula for chi square for this type of problem is

$$\begin{aligned}\chi^2 &= \frac{(a-d)^2}{a+d} \\ &= \frac{(33-15)^2}{33+15} \\ &= \frac{(18)^2}{48} = \frac{324}{48} \\ &= 6.76\end{aligned}$$

In Chapter 12, we found a z of 2.60 for this problem. This squared is equal to 6.76, which it should be, since with 1 *df*, z^2 is equal to chi square.

Testing Goodness of Fit. Another use of chi square is in testing goodness of fit. This is testing a set of data to see if the data are normally distributed. In Chapter 7 we learned how to normalize a set of data. The student who wishes to review at this point is referred to Table 7.2 and the discussion associated with it. In Table 7.2 we set up the expected frequencies for the distribution in the table. Both the observed and expected frequencies from Table 7.2 are reproduced in the first two columns of Table 14.5. It should be apparent by now that anytime that we have a set of observed and a set of expected frequencies we can apply the chi square test.

TABLE 14.5. Testing Goodness of Fit

	(1) O	(2) E	(3) O	(4) E	(5) $O - E$	(6) $(O - E)^2$	(7) $\frac{(O - E)^2}{E}$
110–119	2	2	10	7	3	9	1.286
100–109	8	5					
90–99	12	12	12	12	0	0	.000
80–89	16	19	16	19	3	9	.474
70–79	22	25	22	25	3	9	.360
60–69	36	28	36	28	8	64	2.286
50–59	27	26	27	26	1	1	.039
40–49	13	20	13	20	7	49	2.450
30–39	12	12	12	12	0	0	.000
20–29	8	6	12	10	2	4	.400
10–19	3	2					
0–9	1	2					
	$\Sigma = 160$						$\chi^2 = 7.295$

In columns (1) and (2) of Table 14.5 we have the observed and expected frequencies. In columns (3) and (4) are the same frequencies, this time with the frequencies in the extreme class intervals combined so that none of the expected frequencies is less than 5. In column (5) are shown the differences between each O and E, in column (6) the square of these differences, and in column (7) the square of the differences divided by the expected frequencies. These are then summed and a chi square of 7.295 results.

The null hypothesis that we have here is that the distribution of observed scores does not depart from normal. Basically the number of degrees of freedom for this situation is the number of intervals minus three. We placed three restrictions upon our data when we normalized them. We noted that the best-fitting normal curve for a set of data has the same mean, standard deviation, and number of cases as the original data. One degree of freedom was lost for each of these restrictions. In our problem we reduced the number of class intervals to 9 when we combined those in the tails having small frequencies. For these data it follows that $df = 6$, that is, 9 categories minus 3. In the chi square table we find for $df = 6$ that the 5 percent value of chi square is 12.592. Since our chi square value is smaller than this, the null hypothesis stands, meaning that, as far as we know, our original data do not depart significantly from that of the normal curve.

Exercises

1. A student responds to a 70-item true-false test by guessing the responses to every item. He obtains a score of 45. Does this differ significantly from what would be expected by chance?
2. One hundred cards were pulled from a well-shuffled deck of ordinary playing cards, each card being replaced after being drawn and the deck then reshuffled. Do the following results deviate from what we would expect by chance?

Red card	38
Black card	62
	100
Picture card	16
Number card	84
	100

3. A job information test was given to attendants in a mental hospital. Below are the responses to two items of attendants rated "good" and "poor" by their supervisors. Do the responses to the two items differ?

Item	*Yes*	*No*
Rated good	72	28
Rated poor	22	4
Item 2		
Rated good	12	88
Rated poor	12	14

4. In an experiment subjects were asked to decide whether a sample of a cola drink was one of two well-known brands. Half of the subjects were given brand one and half were given brand two. Brand one was correctly identified by 79 subjects and brand two by 61. Does this differ from what we would expect by chance?
5. Seven dimes were tossed 128 times, with the following results:

7 heads, 0 tails	2
6 heads, 1 tail	8
5 heads, 2 tails	18
4 heads, 3 tails	32
3 heads, 4 tails	41
2 heads, 5 tails	17
1 head, 6 tails	10
0 heads, 7 tails	0

Are these results different from what would be expected by chance?

6. A study showed that 80 percent of freshmen scoring above a certain point on a physical science orientation test passed freshman chemistry. The next year 600 freshmen were tested, and of those scoring at or above the critical score 440 passed and 160 failed. Does this differ significantly from the original results?

7. Two hundred subjects of both sexes were given tastes of oleo and butter and asked to name the one that they preferred. Do the preferences of the two sexes differ in respect to these staples?

	First Choice Oleo	*First Choice Butter*
Males	34	46
Females	72	48

8. In working a chi square problem the following observed and expected frequencies were obtained. Why should the problem be stopped here?

	O		*E*	
Group 1	10	21	10	16
Group 2	11	16	10	16
Group 3	9	15	10	16

9. A nation-wide sample of 2000 high school students was asked to agree or to disagree with the statement: "People should not be allowed to vote unless they are intelligent and educated." The responses of the two sexes were tabulated separately.

	Agree	*Disagree*
Boys	360	640
Girls	290	710

(a) Set up a table for *E*'s and compute chi square.
(b) Solve the problem using formula 15.3.

10. In the same survey this question was posed: "Do you think that some racial and religious groups should be prevented from living in certain sections of cities?" When the responses were tabulated by region, the following frequencies were observed:

	Agree	*Undecided*	*Disagree*
East	89	79	297
Midwest	118	130	350
South	241	140	248
West	37	59	197

(a) Compute the expected frequencies.
(b) How many degrees of freedom are there for this problem?
(c) Do the responses of Eastern students differ from those of Western students on this question?
(d) Compare the responses of Midwestern students with those of Southern students.

11. A group of legislators was asked this question: "Should there be laws against marriage between persons of different races?"

	Agree	*Undecided, Probably Agree*	*Undecided, Probably Disagree*	*Disagree*
Democrats	37	9	13	35
Republicans	26	3	2	43

By the use of chi square, test the significance of the difference of the responses of the members of the two parties.

15

SINGLE CLASSIFICATION ANALYSIS OF VARIANCE

In the previous chapter we were concerned with the z- and t-tests for testing differences between two groups. In actual practice it often happens that more than two groups are involved in a study. One of the writers carried on a study involving a series of tests given to students in five different colleges of a large university. He was interested in differences in performance of the students in the different colleges on these different tests. Using the z-test he could have taken the colleges two at a time and tested for differences between each two. For the five different colleges this would have amounted to ten different z-tests. Frequently, there are more than five groups in a research study, and the number of comparisons that would have to be made to cover all possible tests would be too cumbersome to carry out. The general formula for determining the number of combinations to be made, taking the groups two at a time, is $\frac{N(N-1)}{2}$, where N is the number of groups. For example, if there are 15 groups, we would have to run 105 separate z-tests.

In addition to the nearly prohibitive number of calculations involved in comparing subsamples one by one, there is another, and more important, limitation. When we analyze our data this way, we ignore the fact that these subsamples exist in a set. The elements of such sets are known to *interact*. This interaction should be taken into account in our analysis. Analysis of variance does not ignore interaction. In more complex statistical designs than we will be able to discuss here, estimates of variance due to interaction are made. These estimates often prove to be extremely important in the interpretation of the statistical analysis.

To avoid these limitations, statisticians have designed the so-called analysis of variance techniques in which all of the data are treated at once and a general null hypothesis of no difference among the means of the various groups is tested. In this simple type of analysis of variance we are concerned with two types of variation.

Suppose that we have IQ scores on five samples of adults. The mean and variance (s^2) of each group is

	Group				
	1	*2*	*3*	*4*	*5*
$\bar{X}$	102	123	100	108	112
s^2	15	12	12	14	10

We can see that the means of the groups vary. This variation of group means from the total or grand mean of all groups is referred to as "between groups" variance.

Now let us suppose that we throw all the IQ scores into one big pot and "mix thoroughly." We can forget for the moment which scores belong in which groups. These scores will vary. The variation of these individual scores is called "within group" variance.

The heart of analysis of variance lies in the following fact: If the groups are not different, the two types of variance will be equal. We can test for the significance of the difference of the two types of variance by use of the F-test.

ASSUMPTIONS UNDERLYING THE USE OF THE ANALYSIS OF VARIANCE

When the analysis of variance technique is used, the following assumptions should be met:

1. The individuals in the various subgroups should be selected on the basis of random sampling from normally distributed populations.
2. The variances of the subgroups should be homogeneous. This assumption of homogeneity of subgroup variance should be tested by Bartlett's test (or some other) whenever analysis of variance is to be used (see Bartlett [**1**] or Edwards [**4, 5**]).
3. The trait being studied should normally be distributed in each of the sample subgroups and in its corresponding population.[1]

MEETING THE ASSUMPTIONS

The three assumptions given are *sufficient* for the logical and mathematical theory underlying single classification analysis of variance. However, strictly speaking, the first is not entirely *necessary* for the derivation of the method.

In the *use* of this method, though, all three assumptions are usually stated. These assumptions are not equally critical. It has been empirically shown that the distribution within the subgroups can be skewed to a rather high degree without affecting the significance test (**2**). The third assumption is not usually tested.

The first assumption is dependent upon the adequacy of the experimental design. This emphasizes the necessity for careful planning prior to the execution of statistical tests.

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[1] If these assumptions cannot be made, tests of significance might be applied, using one of the "distribution-free" techniques discussed in Chapter 18.

The second assumption, that of homogeneity of variance, is very important and is almost always tested before the analysis of variance is made. The most frequently used test of this assumption is one devised by Bartlett **1**. The computational method for performing this test is included in the model worksheet shown later as Table 15.5. Bartlett's test yields a chi square. If the chi square is significant with number of groups -1 degrees of freedom, we have reason to believe that the second assumption has not been met. Other methods have been developed for testing samples with significantly different variances.

In this chapter we shall discuss only the very simplest application of the analysis of variance. Usually a good share of a second and third course in statistics is devoted to the many uses and ramifications of this technique.

TESTING THE DIFFERENCES AMONG SEVERAL MEANS

We shall illustrate the basic solution of an analysis of variance problem by using the three sets of data in Table 15.1. Here we have the scores of 7 individuals in three groups, A, B, and C, to a short test. In the three columns at the right, we have the squares of each of these scores.

TABLE 15.1. Example of Single Classification Analysis of Variance

Group A X	Group B X	Group C X	Group A X^2	Group B X^2	Group C X^2
12	18	6	144	324	36
18	17	10	324	289	100
16	16	18	256	256	324
8	18	4	64	324	16
6	12	6	36	144	36
12	17	12	144	289	144
10	10	14	100	100	196
$\Sigma X = 82$	108	70	$\Sigma X^2 = 1068$	1726	852
$X = 11.71$	15.43	10			
$X_T = 12.38$					

The Total Sum of Squares. The total sum of squares could be obtained by finding the mean of the 21 scores, taking the deviation of each score from this mean, and squaring and summing these squared deviations. It may be recalled that we can obtain the sum of the squares by the use of the following equation:

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{N} \tag{15.1}$$

This would mean

$$[(12)^2 + (18)^2 + (16)^2 \ldots + (12)^2 + (14)^2] - \frac{(260)^2}{21}$$

Or from the data in Table 15.1

$$\Sigma x^2 = 1068 + 1726 + 852 - \frac{(82 + 108 + 70)^2}{21}$$
$$\Sigma x^2 = 3646 - \frac{(260)^2}{21}$$
$$\Sigma x^2 = 3646 - \frac{67600}{21}$$
$$\Sigma x^2 = 3646 - 3219$$
$$\Sigma x^2 = 427$$

The "Between" Sum of Squares. The sum of the squares between the various groups can be found by taking the mean of each group, getting its deviation from the total mean, squaring this deviation, and then multiplying each of these by the number of individuals in each group, as follows:

$$\Sigma x^2 = \Sigma(\bar{X} - \bar{X}_T)n \tag{15.2}$$
$$\Sigma x^2 = (11.71 - 12.38)^2(7) + (15.43 - 12.38)^2(7) + (10 - 12.38)^2(7)$$
$$\Sigma x^2 = (-.67)^2(7) + (3.05)^2(7) + (-2.38)^2(7)$$
$$\Sigma x^2 = (.4489)(7) + (9.3025)(7) + (5.6644)(7)$$
$$\Sigma x^2 = 3.1423 + 65.1175 + 39.6508$$
$$\Sigma x^2 = 107.9$$

A more direct method of obtaining the so-called "between" sum of squares is as follows:

$$\Sigma x^2 = \left[\sum\frac{\Sigma X)^2}{n}\right] - \frac{(\Sigma X_T)^2}{N} \tag{15.3}$$
$$\Sigma x^2 = \left[\frac{(82)^2}{7} + \frac{(108)^2}{7} + \frac{(70)^2}{7}\right] - \frac{(260)^2}{21}$$
$$\Sigma x^2 = \frac{6724}{7} + \frac{11664}{7} + \frac{4900}{7} - \frac{67600}{21}$$
$$\Sigma x^2 = (960.6 + 1666.3 + 700) - 3219$$
$$\Sigma x^2 = 3326.9 - 3219$$
$$\Sigma x^2 = 107.9$$

The "Within" Sum of Squares. To obtain the "within" sum of squares we could, group by group, take each score, subtract it from the mean of the group, square each score, and sum.

Group A: $(12 - 12.38)^2 + (18 - 12.38)^2 + \ldots + (10 - 12.38)^2$

This is repeated for the other two groups, and then the three sums are summed. However, this is a rather tedious method and we obtain the identical results as follows:

For Group A:

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{n} \tag{15.4}$$
$$\Sigma x^2 = 1068 - \frac{(82)^2}{7}$$
$$\Sigma x^2 = 1068 - 960.6$$
$$\Sigma x^2 = 107.4$$

For Group B:

$$\Sigma x^2 = 1726 - \frac{(108)^2}{7}$$
$$\Sigma x^2 = 1726 - 1666.3$$
$$\Sigma x^2 = 59.7$$

For Group C:

$$\Sigma x^2 = 852 - \frac{(70)^2}{7}$$
$$\Sigma x^2 = 852 - 700$$
$$\Sigma x^2 = 152$$

Summing for all three groups:

$$\Sigma x^2 = 107.4 + 59.7 + 152$$
$$\Sigma x^2 = 318.1$$

The "within" sum of squares added to the "between" sum of squares should total the "total" sum of squares.

$$319.1 + 107.9 = 427$$

It follows that the "within" sum of squares can be obtained directly by subtracting the "between" sum of squares directly from the sum of squares of the total instead of going through the process described above.

Degrees of Freedom. Since there are 21 cases in the problem that we are working, we have $N - 1$ or 20 *df*. In Group A there are 7 cases; hence there are 6 *df* for this group and, since in this problem the number of cases is the same in each problem, there are 6 *df* in each of the other groups. So far we have accounted for 18 of the total number of degrees of freedom. We have three groups. Then it follows that there are 2 *df* for the groups. To generalize:

df for total groups = number of cases in total (N) minus 1.

df for groups "between" = number of groups (k) minus 1.

df for groups "within" = sum of the number of cases within each subgroup (n) minus 1. $(n_1 - 1) + (n_2 - 1) \ldots$

The Analysis of Variance. The usual technique at this point is to set up a table similar to Table 15.2. Into this table we enter in the appropriate column the number of degrees of freedom, the sum of the squares for each of

TABLE 15.2. Analysis of Variance for the Data in Table 15.1

Source of Variation	*df*	Sum of Squares	Mean Square
"Between" Groups	2	107.9	53.95
"Within" Groups	18	319.1	17.72
Total	20	427	

the three categories, and in the last column the so-called mean square values. These mean squares are obtained by dividing each of the sum of squares by its respective number of degrees of freedom. Such a ratio or division results in variances. The "between" and the "within" mean squares are then two estimates of the population variance.

The F-Test. We made an F-test previously when we were testing the difference between two variances to see whether or not we should go ahead and pool them. The analysis of variance table is evaluated by making the following F-test:

$$F = \frac{\text{mean square for "between" groups}}{\text{mean square for "within" groups}} \tag{15.5}$$

$$F = \frac{53.95}{17.72}$$

$$F = 3.04$$

F-ratios are interpreted by use of the F-Table (Table V of the Appendix). This table is entered across the top with the number of degrees of freedom for the greater mean square and for the number of degrees of freedom in the lesser mean square on the left-hand side. For this problem we go over to 2 and down to 18. At that place we observe that the value of F needed for significance at the 5 percent point is 3.55. Since our value is lower than this, the null hypothesis of no difference between the three groups stands.

There are times when the value of the F-ratio will be less than 1. There is no point in solving for the value of such a ratio as all such ratios are not significant.

THE ANALYSIS OF VARIANCE WITH ONLY TWO GROUPS

The analysis of variance may be applied to only two groups. When this is done, a relationship of the F-ratio and the t-ratio becomes apparent. In Table 15.3 are two sets of scores representing the responses of two groups to a short test. In the table are shown the various sums and means.

TABLE 15.3. Analysis of Variance for Two Groups

X	X_1	X^2	$X_1{}^2$
22	12	484	144
18	16	324	256
24	10	576	100
22	10	484	100
16	4	256	16
18	6	324	36
13	17	169	289
18	14	324	196
19	14	361	196
22	10	484	100
$\Sigma X = 192$	113	$\Sigma = 3786$	1433
$\bar{X} = 19.2$	11.3		

The sum of the squares is computed as follows:

Sum of squares for total:

$$\Sigma x^2 = 5219 - \frac{(305)^2}{20}$$

$$\Sigma x^2 = 5219 - 4651.25$$

$$\Sigma x^2 = 567.75$$

The "between" sum of squares:

$$\Sigma x^2 = \frac{(192)^2}{10} + \frac{(113)^2}{10} - \frac{(305)^2}{20}$$

$$\Sigma x^2 = (3686.4 + 1276.9) - 4651.25$$

$$\Sigma x^2 = 4963.3 - 4651.25$$

$$\Sigma x^2 = 312.05$$

The "within" sum of squares:

Within sum of squares = (total sum of squares) − (between sum of squares)

$$\Sigma x^2 = 567.75 - 312.05 = 255.7$$

The analysis of variance table is set up as shown in Table 15.4 and the F-test is made. This time we find that the F-ratio is equal to 21.96.

Suppose that we next compute a t-ratio for the above data. We shall use the following formula for t, the formula for small samples:

$$t = \frac{\bar{X} - \bar{X}_1}{\sqrt{\frac{\Sigma x^2 + \Sigma x_1{}^2}{(N_1 + N_2) - 2}\left(\frac{1}{N_1} + \frac{1}{N_2}\right)}}$$

$$t = \frac{19.2 - 11.3}{\sqrt{\frac{99.6 + 156.1}{(10 + 10) - 2}\left(\frac{1}{10} + \frac{1}{10}\right)}}$$

$$t = \frac{7.9}{\sqrt{\frac{255.7}{18}[(.1) + (.1)]}}$$

$$t = \frac{7.9}{\sqrt{(14.20)(.2)}}$$

$$t = \frac{7.9}{\sqrt{2.84}}$$

$$t = \frac{7.9}{1.685}$$

$$t = 4.69$$

Both this F of 21.96 and the t of 4.69 are significant beyond the 1 percent level of confidence. Both techniques lead to the same conclusion. It may

TABLE 15.4. Analysis of Variance for the Data in Table 15.3

Source of Variation	df	Sum of Squares	Mean Square
"Between" groups	1	312.05	312.05
"Within" groups	18	255.7	14.21
Total	19	567.75	

be noted that when df for "between" groups $= 1$, $\sqrt{F} = t$ or, putting it the other way around, $t^2 = F$.

$$\sqrt{21.96} = 4.69 = t$$

t-Tests Made After F-Tests. In the problem worked first in this chapter no significant differences were found among the three groups when the F-test was applied to the data. Suppose that this F had been significant. Should we then proceed and make a series of t-tests to see just what groups differ from each other? In the statistical literature there tends to be confusion on the answer to this question. In general, it appears that it is best not to. One might look at the various means and carry on further research using the observed differences that appear to be large enough to be significant.

Tukey (**11**) has come up with an exact answer to this problem of testing the differences between individual means when the F is significant. It involves making three tests, one for the "significant gap," one for the "straggler," and one for "excessive variability." These three tests are not difficult, but they are rather long. The student who is interested is referred to Tukey (**17**) or Edwards (**5**).

A COMPUTATIONAL FORM

A worksheet for computing single classification analysis of variance is shown in Table 15.5. This form includes Bartlett's test for the homogeneity of variance assumption.

In this table the scores on a measure of social class of 971 high school students are tested against the following responses of the students to the question "How do you feel about going to college?"

1. It would be a waste of time.
2. I don't have the ability.
3. I would go chiefly to gain more earning power in terms of pay.
4. I would go chiefly to enjoy social contacts in college and have a good time.
5. I would go chiefly to prepare for more advanced training such as law and medicine.
6. I would go to gain knowledge for its own sake.

Computations from the raw scores are entered in the table in columns (a), (c), and (d). Then, by merely filling in the remaining blanks, as the headings indicate, the analysis is completed.

TABLE 15.5. Worksheet for Computing Single Classification Analysis of Variance

	a	b	c	d	e	f	g	h	i
	n	$n-1$	ΣX	ΣX^2	$\frac{c^2}{a}$	$d-e$	$\frac{f}{b}$	$\log g$	bh
Waste of time	63	62	375	3,079	2,232	847	13.661	1.13548	70.39976
Don't have ability	116	115	557	4,009	2,675	1,334	11.600	1.06446	122.41290
More earning power	196	195	999	6,805	5,092	1,713	8.785	0.94374	184.02930
Social contacts	49	48	295	2,459	1,776	683	14.229	1.15317	55.35216
Prepare for more training	300	299	1,808	14,402	10,896	3,506	11.726	1.06915	319.67585
Gain knowledge	247	246	1,460	11,226	8,630	2,596	10.553	1.02337	251.74902
Total	971	965	5,494	41,980	31,301	10,679	70.554		1003.61899

Other Computations

j	$(\Sigma c)^2$	30,184,036
k	$\frac{j}{\Sigma a}$	31,086
l	$\frac{\Sigma f}{\Sigma b}$	11.0663
m	$\log l$	1.04400

Bartlett: $\chi^2 = 2.3026\,[(\Sigma b)\,m - \Sigma i)] = 8.844$

		SS	*df*	*MS*	*F*
Total	$\Sigma d - k$	10,894			
"Within"	$\Sigma d - \Sigma e$	10,679	965	11.07	
"Between"	$\Sigma e - k$	215	5	43.00	3.88

First the Bartlett test of homogeneity of variance is made. The constant, 2.3026, multiplied by the entries from the table yields a chi square. In this example, the chi square of 8.844 is not significant. This means that variances of the social class scores of students choosing the alternatives can be assumed to have come from the same population distribution.

We can now legitimately make the *F*-test. The sums of squares and degrees of freedom are entered into the final table as the headings show us. The "within mean square" and "between mean square" are found by dividing their corresponding sum of squares by their degrees of freedom. Finally, the value of *F*, the ratio of the two mean squares, is computed. In this example, this ratio equals 3.88. By comparing this value with the *F*-table (Table V in Appendix), we find that the *F* is significant. This allows us to conclude that the students choosing different alternatives have significantly different social class scores.

Exercises

1.

X	*Y*	*X*	*Y*
40	29	22	27
46	38	20	45
35	20	31	17
17	27	18	26
11	28	22	29

(a) Using an *F*-test, test the difference between the means of groups *X* and *Y*.
(b) Test the significance of the difference between these two means using a *t*-test.

2. The following data represent the scores of ten students in each of five schools of a university on the *Ohio Psychological Examination*. By means of analysis of variance, test the difference between the means.

Liberal Arts	*Business Administration*	*Home Economics*	*Fine Arts*	*Engineering*
106	79	133	50	109
120	91	78	60	129
81	105	113	112	111
92	127	40	97	77
102	71	103	80	102
121	115	80	62	130
86	95	113	116	125
131	115	90	134	75
62	62	84	98	93
122	52	113	112	82

3. If the *F*-test in a single classification analysis of variance is not significant, can we say that the means of the population from which the samples were drawn are equal?
4. What do we call the risk we take of being wrong in rejecting a null hypothesis?

5. The enrollment of the grades in an elementary school is as follows:

First	75
Second	86
Third	72
Fourth	68
Fifth	61
Sixth	58
Seventh	53
Eighth	43

(a) Can we test the significance of the difference between the enrollments by analysis of variance?
(b) Can we test by other means?

6. Given the following four random samples, test the hypothesis that they are from the same population.

Sample A	*Sample B*	*Sample C*	*Sample D*
15	24	20	25
20	22	22	18
26	20	30	16
26	21	27	32
24	34		24
	18		

7. Three types of advertising were tested for their effectiveness. Their "effectivity" mean scores are:

$$M_1 = 80 \qquad M_2 = 45 \qquad M_3 = 63$$

An analysis of variance was performed and the means were found to differ significantly.
(a) Do we know that Type 1 is better than Type 2? Type 1 is better than Type 3?
(b) If you are the advertiser and Type 1 costs one-third more than Type 3, what would you do?
(c) How might you use sales records?

16

OTHER CORRELATIONAL TECHNIQUES

In Chapter 8, the entire chapter was devoted to the most widely used correlation coefficient, the Pearson r. When the data that we wish to correlate meet the assumptions basic to the computation of the Pearson r and when the two variables are continuous, it is the best statistic to use. However, there are times when the relationship between two variables is not linear, when one or both of the variables are not continuous, or when the number of pairs of measurements is too small or not randomly collected so that the Pearson r is not applicable. In this chapter we shall study some of the special correlation techniques which may be applied to some of the situations just described.

THE POINT-BISERIAL r_{pb}

There are circumstances, especially in the field of test construction and test validation, where one of the variables is a continuous one and the other is conceived as being a dichotomy. In the usual scoring of items, the procedure is to mark the item either right or wrong. This right-wrong scoring is regarded as being a true dichotomy. In item analysis work (see Chapter 17) the testmaker is usually concerned with how well the item is separating the good students from the poor ones. The ability of an item to discriminate between the two groups is frequently measured by a correlation coefficient. The usual procedure is to correlate the responses of each item with the total test score.

This is done by setting up a table like Table 16.1. In the first column appears an interval grouping for the total test scores. Then the papers are taken one by one and the first item on the test is examined to see whether or not the subject answered it correctly. Suppose that the first paper has a score of 73 and that this student answered the item correctly. Then, a tally is placed in the "Right" column, column (2). Each paper is taken, and the

TABLE 16.1. Worksheet for the Point-Biserial and Biserial r

(1)	(2) f_P Right	(3) f_W Wrong	(4) f_t	(5) x'	(6) $f_t x'$	(7) $f_t x'^2$	(8) fx'
70–74	3	0	3	5	15	75	15
65–69	6	1	7	4	28	112	24
60–64	6	2	8	3	24	72	18
55–59	5	4	9	2	18	36	10
50–54	6	2	8	1	8	8	6
45–49	7	6	13	0	0	0	0
40–44	6	8	14	−1	−14	14	−6
35–39	3	6	9	−2	−18	36	−6
30–34	3	9	12	−3	−36	108	−9
25–29	1	4	5	−4	−20	80	−4
20–24	0	12	12	−5	−60	300	0
	$\Sigma = 46$	$\Sigma = 54$	$\Sigma = 100$		$\Sigma = -55$	$\Sigma = 841$	$\Sigma = 48$

response of the first item is tallied into either column (2) or (3). Actually, what we have is a scatterplot with the variable on the Y-axis continuous, just as with the Pearson r, but with the variable on the X-axis reduced to two categories.

Column (4) is the total number of frequencies falling in each of the intervals. The sum of this is 100 which is the sum of the number who responded correctly and the number who responded incorrectly to the item. We now proceed to compute the mean and standard deviation of the total scores. Columns (5), (6), and (7) are set up in the usual way and columns (6) and (7) are summed. We also need the mean of the individuals getting the item correct. For this we set up column (8) which is the product of the values in columns (2) and (5).

A formula for the point-biserial is as follows:

$$r_{pb} = \frac{\bar{X}_p - \bar{X}_t}{s_t}\sqrt{\frac{p}{q}} \tag{16.1}$$

where

$\bar{X}_p$ = the mean score of those answering the item correctly.

$\bar{X}_t$ = the mean of the total test scores.

s_t = the standard deviation of the test.

p = the proportion of the total group answering the item correctly.

$q = 1 - p$.

We first solve for the two means.

$$\bar{X}_p = 47 + \frac{48}{46}(5) \qquad \bar{X}_t = 47 + \frac{-55}{100}(5)$$

$$\bar{X}_p = 47 + 5.2 \qquad \bar{X}_t = 47 + (-2.75)$$

$$\bar{X}_p = 52.2 \qquad \bar{X}_t = 44.2$$

The standard deviation must also be computed.

First, the Σx^2:

$$\Sigma x^2 = \left[841 - \frac{(-55)^2}{100}\right](5)^2$$

$$= (841 - 30.25)(25)$$

$$= 20268.75$$

$$p = \frac{46}{100} = .46$$

Then, s_t:

$$s_t = \sqrt{\frac{20268.75}{99}}$$

$$= \sqrt{204.73}$$

$$= 14.3$$

Substituting these values in equation 16.1, we have

$$r_{pb} = \frac{52.2 - 44.2}{14.3}\sqrt{\frac{.46}{.54}}$$

$$= \frac{8}{14.3}\sqrt{.851851}$$

$$= .559(.923)$$

$$= .516 \text{ or } .52$$

This coefficient is a product-moment correlation coefficient and is used and interpreted just as the Pearson r is. If the test being analyzed contains a large number of items, the use of this statistic is almost prohibitive from the point of view of time. There are some computing aids which will facilitate such computations, but in item analysis work the point biserial is not used too frequently. In test validation work, the criterion may be "Pass" or "Fail," "Obtained his wings," "Did not obtain his wings," "On probation," or "Not on probation," and the like. The relationship between the test scores and such criteria may be computed by the use of this statistic.

Reliability of the Point-Biserial. The significance of the point-biserial correlation coefficient may be tested by the use of the following t-test and entering the t-table with $N - 2$ degrees of freedom.

$$t = \frac{r_{pb}\sqrt{N - 2}}{\sqrt{1 - r_{pb}^2}} \tag{16.2}$$

For these data:

$$t = \frac{.52\sqrt{98}}{\sqrt{1 - (.52)^2}}$$

$$t = \frac{(.52)(9.8995)}{\sqrt{1 - .2704}}$$

$$t = \frac{4.1477}{.852}$$

$$t = 4.86$$

which with $df = 98$ is significant.

A quick and efficient method of estimating the point-biserial r is to use an abac prepared by Dingman (Figure 16.1). This abac can be used when one variable is divided at the median. The proportion of subjects in the upper criterion group who pass a given item is found on the ordinate. The corresponding proportion from the "low" group is found on the abscissa. The estimated point-biserial r is found at the perpendicular intersection of these values.

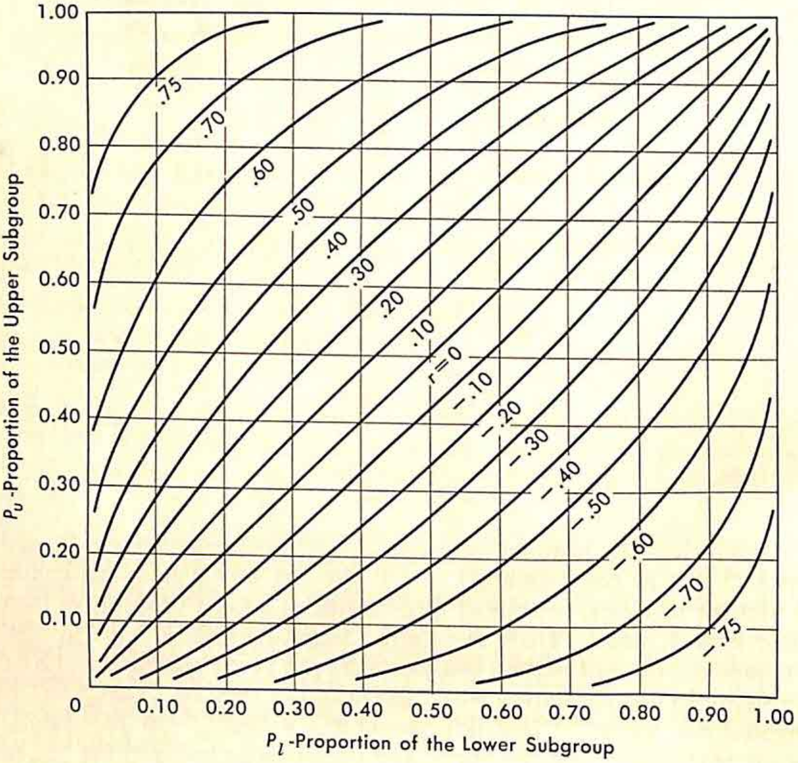


FIGURE 16.1 **An abac for estimates of the point-biserial r when one variable is divided at the median of distribution. (Prepared by Harvey F. Dingman.)**
Source: J. P. Guilford. *Psychometric Methods*. New York: McGraw Hill, 1954. Reprinted by permission of the publisher.

THE BISERIAL CORRELATION COEFFICIENT

Another statistic widely used in item analysis work is the biserial $r(r_b)$. When we use this statistic we have one variable continuous and the other actually continuous but forced into a dichotomy. Here is an example of a forced dichotomy. Passing and failing may be thought of as being on a continuum ranging from those who pass with exceedingly high honors down to those who fail miserably. Passing is made up of groups of individuals from the honor students down to the borderline cases, and failure includes all those who just barely failed down to the utter failures. We reduce this continuum to a "pass-fail" dichotomy, and as this is the usual procedure in test scoring, the biserial r may be used as a measure of the discrimination index of an item.

To find the value of the biserial r, one of the easiest formulas is

$$r_b = \frac{\bar{X}_p - \bar{X}_t}{s_t}\left(\frac{p}{y}\right) \tag{16.3}$$

where all symbols are the same as defined for the point-biserial r except y which is the ordinate obtained from the normal probability table, cutting off above it an area equal to p.

We shall take the data in Table 16.1 and this time solve for the biserial r.

$$r_b = \frac{52.2 - 44.2}{14.3}\left(\frac{.46}{.3969}\right)$$

$$r_b = \frac{8}{14.3}(1.159)$$

$$r_b = .559(1.159)$$

$$r_b = .65$$

The biserial r is an estimation of the product-moment correlation coefficient that would be obtained if both variables were treated as actually being continuous. The standard error of r_b is larger than that of r, but actually the sampling distribution of the statistic is not well known. Both this statistic and the one previously studied are most reliable when the dichotomies are made somewhere near the middle of the distribution. The further that one departs from this, the less reliable the statistic. All papers must be used for these techniques. One must not take the upper 27 percent or 30 percent and contrast these papers with their corresponding percents at the other end of the distribution.

Flanagan's abac provides an efficient method for estimating r_b (Figure 16.2). This device is designed for use when the "upper" and "lower" groups have each been selected to contain approximately 27 percent of the total distribution (see Chapter 17).

THE FOURFOLD COEFFICIENT OR PHI

The phi coefficient is used when each of the variables is reduced to a dichotomy. To illustrate this technique, let us suppose that we are making an item analysis of a test which has been administered to 370 students. For our analysis we are going to use the highest 100 papers and the lowest 100 papers. We set the data up as shown in Table 16.2, with the high and low groups on the side and the score of "Right" or "Wrong" across the top. The number in the high and low groups that answered the item correctly and incorrectly is entered into the cells. Note that the cells are lettered and that

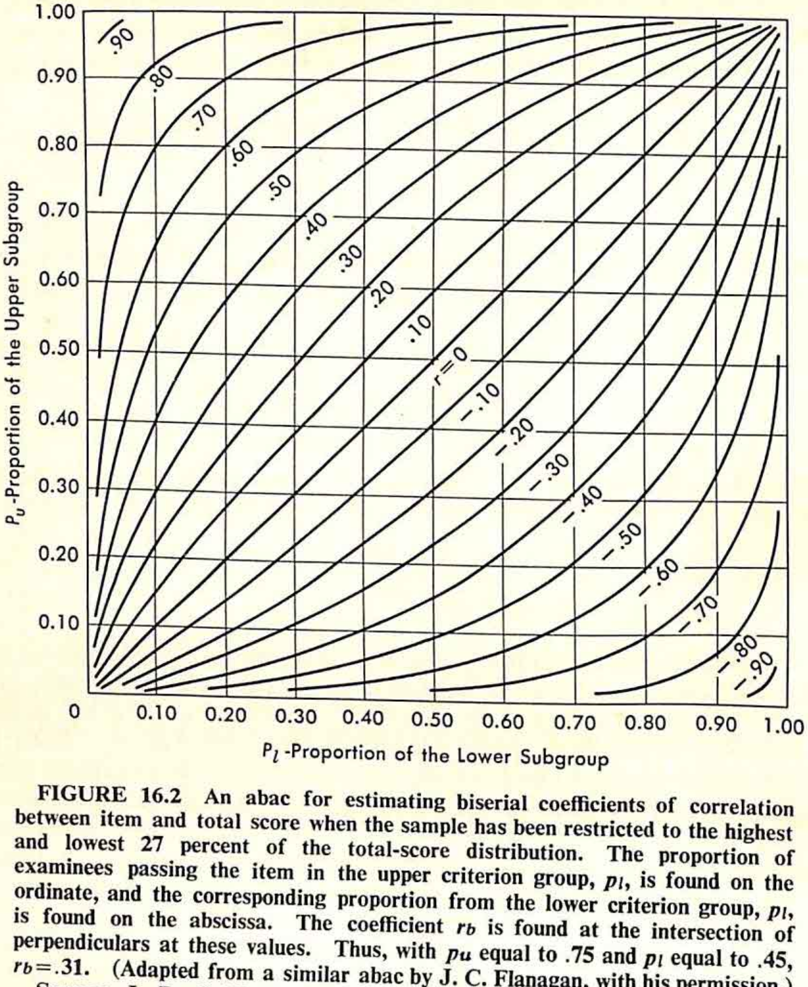


FIGURE 16.2 An abac for estimating biserial coefficients of correlation between item and total score when the sample has been restricted to the highest and lowest 27 percent of the total-score distribution. The proportion of examinees passing the item in the upper criterion group, p_l, is found on the ordinate, and the corresponding proportion from the lower criterion group, p_l, is found on the abscissa. The coefficient r_b is found at the intersection of perpendiculars at these values. Thus, with p_u equal to .75 and p_l equal to .45, $r_b = .31$. (Adapted from a similar abac by J. C. Flanagan, with his permission.)

Source: J. P. Guilford. *Psychometric Methods*. New York: McGraw Hill, 1954. Reprinted by permission of the publisher.

TABLE 16.2. Responses of the Highest and Lowest One Hundred Papers to a Test Item

	Right	Wrong	
Upper 100	70 a	30 b	100 k
Lower 100	30 c	70 d	100 l
	100 m	100 n	200 N

marginal values have been computed and given letters. The formula for the phi coefficient is

$$\Phi = \frac{(ad - bc)}{\sqrt{(k)(l)(m)(n)}} \tag{16.4}$$

where the various letters are the frequencies as shown in Table 16.2.

Solving for these data, we have

$$\begin{aligned}\Phi &= \frac{(70)(70) - (30)(30)}{\sqrt{(100)(100)(100)(100)}}\\ &= \frac{4900 - 900}{10{,}000}\\ &= \frac{4{,}000}{10{,}000}\\ &= .40\end{aligned}$$

Guilford (**6**) has shown that when there is an equal number in both parts of the split, phi can be obtained by the following formula:

$$\Phi = \frac{p_u - p_1}{2\sqrt{pq}} \tag{16.5}$$

where

p_u = proportion in the upper group answering the item correctly.
p_1 = proportion correct in the lower group answering the item correctly.
p = proportion in the two groups responding correctly to the item.
$q = 1 - p$.

Using this formula for these data, we have

$$\begin{aligned}\Phi &= \frac{.70 - .30}{2\sqrt{(.50)(.50)}}\\ &= \frac{.40}{2(.5)}\\ &= \frac{.40}{1} = .40\end{aligned}$$

Guilford also presents an abac for graphic estimation of the phi coefficient when one variable has an even division of cases in two categories. This abac is presented in Figure 16.3.

Testing the Significance of Phi. The student has probably been struck by the similarity of equation 16.4 with the equation for chi square when data are set up into a fourfold table. The two equations are very similar and the relationship between these two statistics can be given by

$$\chi^2 = N\Phi^2 \tag{16.6}$$

Because of this relationship, the phi coefficient may be tested for significance by the use of chi square. For any number in two groups, we can set up the

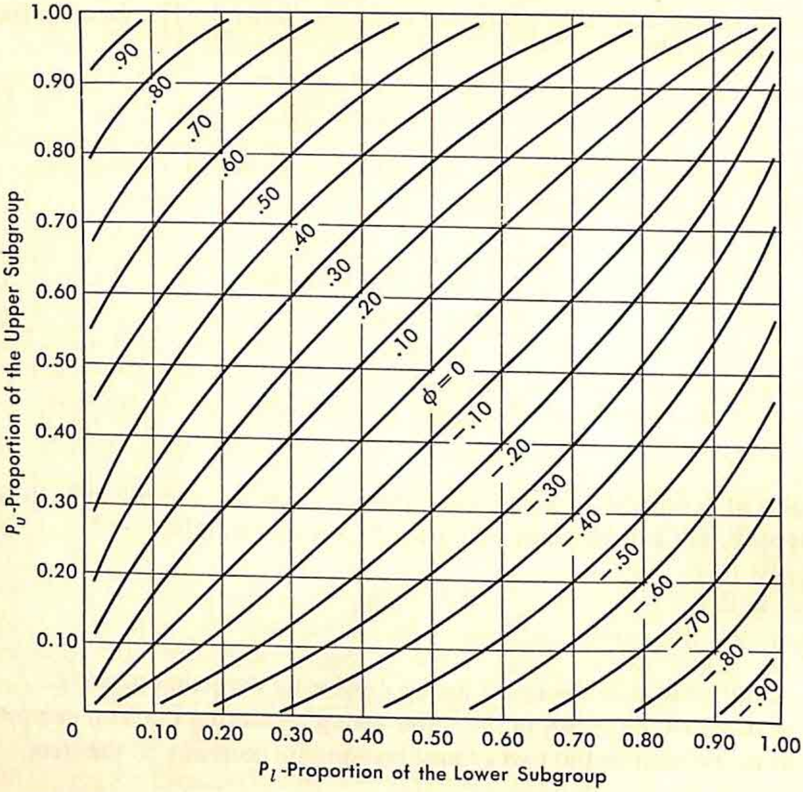


FIGURE 16.3 An abac for graphic estimates of the phi coefficient when one variable has an even division of cases in two categories. Used similarly to the abac in Figure 16.1.

Source: J. P. Guilford. *Psychometric Methods*. New York: McGraw Hill, 1954. Reprinted by permission of the publisher.

1 percent and 5 percent values of phi and use these to evaluate the significance of the phi coefficients for the items of the test being analyzed. Obtaining the 1 percent and 5 percent values of chi square for $df = 1$ from the chi square table, we have

$$\begin{aligned} \chi^2 &= N\Phi^2 \\ 6.635 &= 200\Phi^2 \\ \Phi^2 &= .03318 \\ \Phi &= .18 \end{aligned}$$

$$\begin{aligned} \chi^2 &= N\Phi^2 \\ 3.841 &= 200\Phi^2 \\ \Phi^2 &= .019205 \\ \Phi &= .14 \end{aligned}$$

Since this statistic is related to chi square, it, like chi square, is based upon no assumptions concerning the shape of the distribution of scores. In the past it was assumed that both of the dichotomies were true dichotomies. Such an assumption is not necessary, and the technique may be applied to two forced dichotomies. Because of the nature of the statistic, it favors items which have difficulty values of 50 percent.

THE COEFFICIENT OF CONTINGENCY

When data have been arranged in more than a 2 × 2 table, the coefficient of contingency may be used as a measure of relationship. If the chi square value for a contingency table proves to be significant (meaning that the null hypothesis of no relationship has been rejected), a measure of relationship can be obtained using the contingency coefficient (C). The formula for this coefficient follows:

$$C = \sqrt{\frac{\chi^2}{N + \chi^2}} \tag{16.7}$$

This formula must not be applied to *any* chi square value but should be used only when the data to be tested can be arranged into a meaningful order. C is one of the few statistics applicable to nominal or qualitative data.

THE TETRACHORIC CORRELATION COEFFICIENT

The tetrachoric correlation coefficient also involves setting up data into a fourfold table. An assumption associated with this statistic is that both of the variables are continuous ones that have both been forced into a dichotomy. In the past the tetrachoric coefficient was obtained by solving a quadratic equation for r_t to the second power. Since the use of these equations is laborious, various computing devices have been developed to aid in the computation of the statistic. The one that we shall demonstrate here is that proposed by Davidoff and Goheen (Table VIII in the Appendix).

Suppose that we have the responses of 100 individuals to a test item. A table such as that shown in Table 16.3 is set up and the values in the various cells tabulated. Our papers were divided into the top 50 and the bottom 50, and then the right and wrong answers for each group were counted. Notice again that the cells are lettered in the usual fashion. Davidoff and Goheen's table (Table VIII in the Appendix) is entered with the value ad/bc and from the table the estimation of the tetrachoric coefficient is obtained. For these data:

TABLE 16.3. Responses of One Hundred Individuals to a Test Item, the Data Being Arranged for the Solution of the Tetrachoric Correlation Coefficient

	Right	Wrong	
Upper 50	20 *a*	30 *b*	50
Lower 50	10 *c*	40 *d*	50
	30	70	100

$$\frac{(20)(40)}{(30)(10)} = \frac{800}{300} = 2.67$$

Going into the tables with a value of 2.67 leads to an r_t of .37. Solving the equation for these data resulted in an r_t of .36. Both of these, however, are approximations, for this quadratic equation is only solved for r_t to the second power, all higher powers being ignored.

If *ad* is less than *bc*, we use the ratio *bc*/*ad* for entering the tables. The larger of the two products is always placed in the numerator. This table works best when both variables have been dichotomized on the basis of a 50–50 split. As a statistic r_t is less reliable than the Pearson r but as its standard error is complicated, it will not be discussed here.

RANK CORRELATIONS

Many times data are collected in the form of ranks. Sometimes one variable may be in this form and the other comprised of measurement data. There are other times when measurement data are reduced to ranks, for example, when samples are small and when the assumptions for parametric statistics cannot be met. Two major coefficients can be applied to such data, the Spearman rank-order correlation coefficient and Kendall's coefficient of concordance.

SPEARMAN RANK-ORDER CORRELATION COEFFICIENT (RHO)

This is the most widely used of the rank correlational methods. It is particularly well-suited to situations where the number of cases is 25 or 30 or less. It is also much easier and faster to compute than the Pearson r.

To illustrate the computation of the rank-order correlation coefficient, we shall use the data in Table 16.4. In this table the scores of 17 boys on the

TABLE 16.4. Calculation of the Spearman Rank-Order Correlation Coefficient

Individual	Minn. Paper Form Board	Otis Self-Administering Test	R_1	R_2	D	D^2
1	60	60	1	2	1	1
2	54	68	2	1	1	1
3	53	40	3	11.5	8.5	72.25
4	49	52	4.5	3	1.5	2.25
5	49	51	4.5	4.5	0	0
6	47	38	6	14	8	64
7	46	51	7	4.5	2.5	6.25
8	45	32	9	17	8	64
9	45	39	9	13	4	16
10	45	41	9	10	1	1
11	43	50	11	6	5	25
12	41	48	12	7.5	4.5	20.25
13	39	36	13	16	3	9
14	38	48	14	7.5	6.5	42.25
15	32	40	15.5	11.5	4	16
16	32	46	15.5	9	6.5	42.25
17	30	37	17	15	2	4
						$\Sigma = 386.50$

Minnesota Paper Form Board and on the *Otis Self-Administering Test of Mental Ability* are given. To obtain the coefficient, we go through the following steps:

1. Take the first column of scores (*Minnesota Paper Form Board*) and rank them, giving the high score a rank of 1. Since these scores have been arranged from high to low this is easy. However, notice that we have two individuals tied with scores of 49. These two 49's occupy ranks 4 and 5. These two ranks are averaged and both individuals are assigned this average rank which in this case is 4.5. Too many such ties will affect the size of the correlation coefficient; but usually there are not enough to justify the formulas that are available to correct these ties.
2. Rank the second set of scores again, assigning the rank of 1 to the highest score.
3. Obtain the difference between the two sets of ranks. The sign of this difference is of no importance as these differences are squared in the next operation.
4. Square each of these differences and sum this column of squares.
5. Solve for the rank-order correlation coefficient by the use of the following equation:

$$\rho = 1 - \frac{6\Sigma D^2}{N(N^2 - 1)} \tag{16.8}$$

where

N = the number of pairs.

ρ = rho, the rank-order correlation coefficient.

$$\rho = 1 - \frac{6(386.5)}{17(289 - 1)}.$$

$$\rho = 1 - \frac{2319.0}{4896}.$$

$$\rho = 1 - .47.$$

$$\rho = .53.$$

Interpretation of Rho. Rho is a product-moment correlation coefficient for ranked data. For all practical purposes it may be interpreted the same as r. When N is small, special tables are available for testing the significance of rho. If the size of the sample is over 25, the t-test can be made, using the following formula:

$$t = \frac{\rho}{\sqrt{1 - \rho^2}}\sqrt{N - 2} \tag{16.9}$$

This is the same formula used for testing the significance of an r. The student may recall that a table can be used in lieu of this formula for r. The same table may be used in testing the significance of rho when the number of pairs is 20–25 and above.

Summary. In rho we have a coefficient that makes a good substitute for r

when the number of cases is small. Rho is almost useless when N is large, for by the time that all the data are ranked, a Pearson r could have been computed by the scattergram or some other method. This statistic is interpreted in the same manner as r and has approximately the same size standard error.

KENDALL'S COEFFICIENT OF CONCORDANCE, *W*

If one wishes to determine the relationship among three or more sets of ranks, one rank could be selected and a Spearman rho coefficient computed between it and all of the others, and this process could then be continued until a rho coefficient has been obtained between each set of two ranks. Then these rho's could be averaged for an overall measure of relationship.

Kendall, though, has developed a technique and a statistic which makes all of this unnecessary. Suppose that five judges (m) rank the projects of ten individuals (N) in a judging contest, and we wish to determine the overall relationship among the ratings of the five judges. The rankings of these judges have been set up in Table 16.5. First the rankings by the five judges of each of the projects are summed and appear in column (3). Column (3) is summed to give the total sum of the ranks. This can be checked for the total sum of the ranks, as follows:

TABLE 16.5. Calculation of the Coefficient of Concordance, the Data Consisting of the Ranking of Ten Projects by Five Judges

(1) Individual Project	(2) Judges' Ranks					(3) Sum of Ranks	(4) D	(5) D^2
	1	2	3	4	5			
1	2	1	2	3	4	12	15.5	240.25
2	1	3	1	2	2	9	18.5	342.25
3	3	4	4	1	3	15	12.5	156.25
4	5	5	5	5	1	21	6.5	42.25
5	4	2	6	7	6	25	2.5	6.25
6	7	8	3	4	7	29	1.5	2.25
7	6	6	8	6	5	31	3.5	12.25
8	8	7	7	8	9	39	11.5	132.25
9	9	10	10	9	8	46	18.5	342.25
10	10	9	9	10	10	48	20.5	420.25
						$\Sigma = 275$		$\Sigma = 1696.50$

$$\text{Total sum of ranks} = \frac{mN(N+1)}{2}$$

$$= \frac{(5)(10)(11)}{2} = 275 \qquad (16.10)$$

If there were no relationship among the ranks, we should expect the sum of the ranks for each row to be equal. For this case the sum of each would be the average sum of ranks or $^{275}/_{10}$ which equals 27.5. We next obtain the difference of the sum of the ranks of each row from this mean and then square these differences. Then these squares are summed. This work appears in columns (4) and (5) of Table 16.5.

To compute W, we use the following formula:

$$W = \frac{12\Sigma D^2}{m^2(N)(N^2 - 1)} \tag{16.11}$$

$$W = \frac{12(1696.5)}{(25)(10)(100 - 1)}$$

$$W = .82$$

Interpretation of W. The size of this coefficient of concordance indicates that there is high agreement among these five judges in the ranking of the ten projects. Perfect agreement is indicated by a $W = 1$ and lack of agreement by a $W = 0$. The significance of a coefficient of concordance may be tested by the use of tables developed by Kendall. For $m = 5$ and $N = 10$, we find that the W here computed is highly significant (see Table XIII, in the Appendix).

NONLINEAR RELATIONSHIPS

It frequently happens that the relationship between two sets of variables is not linear. When such a relationship occurs, the Pearson r and variations of this r as described in this chapter are inappropriate as measures of relationship. When a relationship is curvilinear something like this happens. As one variable increases, the other may increase up to a point and then begin and continue to decrease as the first variable continues to increase. Examples of this were noted in the chapter on correlation. When r is used as a measure of relationship with data which are nonlinear, the r so calculated is always an underestimate of the real relationship between the two variables. Sometimes the relationship is actually very high, but the resulting r approaches zero. Since it is difficult to tell whether or not data are linear just by looking at them, a scatterplot should always be made. If a deviation from linearity is apparent or even suggested on this scattergram, then the Pearson r should not be used.

The correct coefficient to use when the relationship between two sets of data is curvilinear is the correlation ratio or the eta coefficient. We shall illustrate the computation of this coefficient by using the data in Table 16.6, which consists of the scores of 45 high school students on two scales, the *Otis Self-Administering Test of Mental Ability* and the *Rotter Incomplete Sentence Blank*. In running this study the research worker hypothesized that the brightest and dullest students had the highest Rotter scores, this being an indication of more adjustment problems. Such an hypothesis assumes that the relationship between the two variables is curvilinear.

Computation of Eta. In solving for eta, the first thing to do is to set up

TABLE 16.6. Scores of Forty-Five High School Students on the *Otis Self-Administering Test of Mental Ability* and the *Rotter Incomplete Sentence Blank*

Otis	Rotter	Otis	Rotter
104	98	85	123
101	104	95	124
117	106	100	125
101	106	109	125
119	108	126	127
91	111	125	128
102	111	112	129
91	111	109	129
122	112	98	131
101	113	95	133
101	114	97	133
87	116	102	134
97	116	110	134
104	117	98	134
89	117	102	136
103	118	102	136
106	119	86	136
86	119	94	137
94	119	87	140
108	119	116	141
93	121	105	148
91	122	101	148
		115	151

SOURCE: Data from B. Junker. A comparison between performance on an incomplete sentence blank and two intelligence tests. Master's thesis, Purdue University, 1954.

a scatterplot. Table 16.7 illustrates a scatterplot for the data in Table 16.6. Here we have the Otis scores on the Y-axis and the Rotter values on the X-axis. The tallies are entered, and the y-values in the right-hand columns are computed and summed.

The most direct approach to the solution of eta is to define eta squared as the ratio of the sum of the squares for the "between" columns for variable-Y to the total sum of the squares for variable Y.

$$\eta_{yx}{}^2 = \frac{\Sigma y_b{}^2}{\Sigma y_t{}^2} \tag{16.12}$$

and eta is

$$\eta_{yx} = \sqrt{\frac{\Sigma y_b{}^2}{\Sigma y_t{}^2}} \tag{16.13}$$

The total sum of the squares for y is readily available from the data on the scattergram.

$$\Sigma y_t{}^2 = \Sigma fy'^2 - \frac{(\Sigma fy')^2}{N}$$

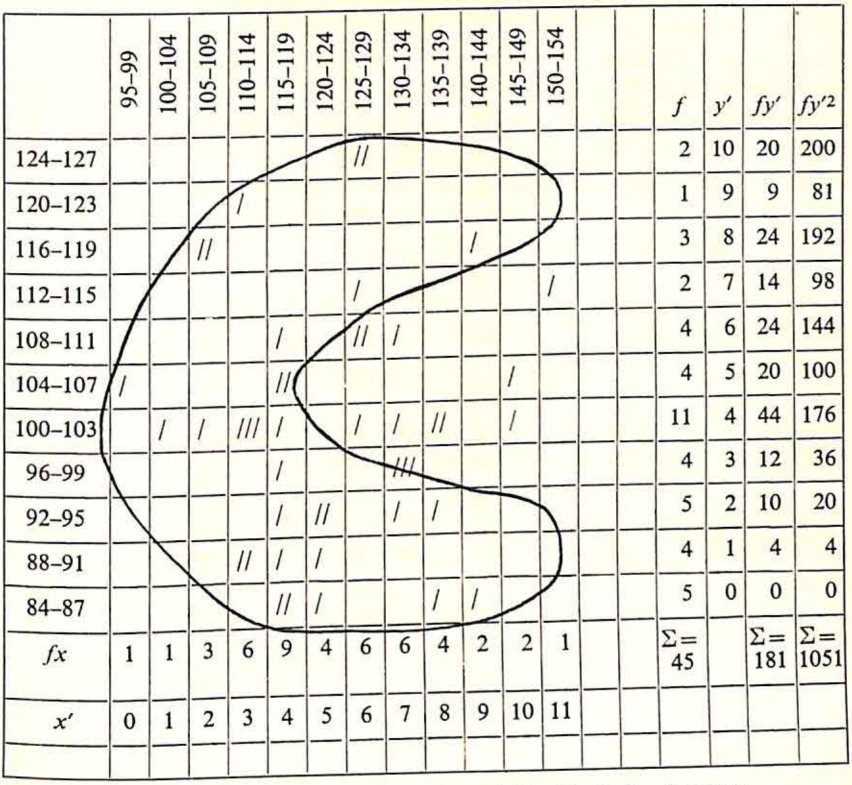
$$\Sigma y_t{}^2 = 1051 - \frac{(181)^2}{45}$$

$$\Sigma y_t{}^2 = 1051 - 728$$

$$\Sigma y_t{}^2 = 323$$

TABLE 16.7. Scatterplot of *Otis Self-Administering Test* Scores and *Rotter Incomplete Sentence Blanks* for Forty-Five Individuals

X-Axis—*Rotter Incomplete Sentence Blank*



Y-Axis—*Otis Self-Administering Test*

	95–99	100–104	105–109	110–114	115–119	120–124	125–129	130–134	135–139	140–144	145–149	150–154			*f*	*y′*	*fy′*	*fy′*^{2}
124–127							//								2	10	20	200
120–123				/											1	9	9	81
116–119			//							/					3	8	24	192
112–115							/					/			2	7	14	98
108–111					/		//	/							4	6	24	144
104–107	/				//						/				4	5	20	100
100–103		/	/	///	/		/	/	//		/				11	4	44	176
96–99					/			///							4	3	12	36
92–95					/	//		/	/						5	2	10	20
88–91				//	/	/									4	1	4	4
84–87					//	/			/	/					5	0	0	0
fx	1	1	3	6	9	4	6	6	4	2	2	1			Σ= 45		Σ= 181	Σ= 1051
x′	0	1	2	3	4	5	6	7	8	9	10	11						

TABLE 16.8. Calculation of the Correlation Ratio for the Data in Table 16.6

(1) Column	(2) f_x	(3) $\Sigma y'$	(4) $(\Sigma y')^2$	(5) $\dfrac{(\Sigma y')^2}{f_x}$
0	1	5	25	25
1	1	4	16	16
2	3	20	400	133.33
3	6	23	529	88.17
4	9	26	676	75.11
5	4	5	25	6.25
6	6	43	1849	308.17
7	6	21	441	73.5
8	4	10	100	25
9	2	8	64	32
10	2	9	81	40.5
11	1	7	49	49
	N = 45	Σ = 181		Σ = 872.03

The calculation of the "between" sum of squares for y is shown in Table 16.8. This "between" sum of squares is determined from the means of the columns taken from the mean of the entire distribution. The first column in Table 16.8 identifies the columns. It consists of the x' values from the bottom of the scattergram. Column (2) is the frequencies of the various columns. In column (3) we find the sum of the y' (deviations from the arbitrary reference point for y) for the frequencies of each of the columns. Column 0 has one frequency with a y' value of 5. The third column, however, has several frequencies, two with a y' value of 8 and one with a y' value of 4, making a sum of 20. In this way all of the values in column (3) are obtained. Each of the values in column (3) is then squared, the square placed in column (4), and then each of these squares divided by the appropriate column frequency. These are summed and the sum of squares for "between" columns is found as follows:

$$\Sigma y_b^2 = \sum\left[\frac{(\Sigma y')^2}{f_x}\right] - \frac{[\Sigma(\Sigma y')]^2}{N}$$

$$\Sigma y_b^2 = 872 - \frac{(181)^2}{45}$$

$$\Sigma y_b^2 = 872 - 728$$

$$\Sigma y_b^2 = 144$$

For these data

$$\eta_{yx} = \sqrt{\frac{144}{323}}$$

$$\eta_{yx} = \sqrt{.4458}$$

$$\eta_{yx} = .666$$

This can all be combined into one formula for eta squared

$$\eta_{yx}{}^2 = \frac{\sum\left[\frac{(\Sigma y^1)^2}{f_x}\right] - \frac{[\Sigma(\Sigma y^1)]^2}{N}}{\Sigma f y'^2 - \frac{(\Sigma f y')^2}{N}} \tag{16.14}$$

When we were dealing with the correlation coefficient, we noted that the correlation between X and Y was the same as that between Y and X and that only one needed to be computed. This is not the same with the correlation ratio, and the other eta coefficient can be determined by inserting the corresponding values in equation 16.14.

Significance of the Correlation Ratio. The significance of the correlation may be tested by the use of the F-test, in which F is the usual ratio of the "between" mean square divided by the "within" mean square. The mean square between columns is obtained by dividing the sum of the squares for the "between" columns by the number of columns in the scatterplot, minus one.

$$\frac{144}{c-1} = \frac{144}{11} = 13.09$$

The "within" sums of squares is first obtained.

$$\Sigma x_t^2 - \Sigma x_b^2 = \Sigma x_w^2$$
$$323 - 144 = 179$$

The mean square for "within" columns is obtained by dividing the sum of the squares for "within" columns by the number of pairs in the sample less the number of columns.

$$\frac{179}{N - c} = \frac{179}{45 - 12} = \frac{179}{33} = 5.42$$

Then the *F*-ratio is computed.

$$F = \frac{13.09}{5.42} = 2.42$$

This *F* is interpreted by going into the *F*-table with $(c - 1)$ degrees of freedom for the "between" mean square and $(N - c)$ *df* for the "within" mean square. In this situation these degrees of freedom are 11 and 33. Entering the table at this point shows that the correlation ratio computed for these data is significant at the 5 percent point. It is interesting to note here that the Pearson *r* for these same data is .003.

Summary. If the data in a scatterplot are linear in their relationship, the value of eta and the Pearson *r* computed for these data would be identical. If the data are curvilinear, eta is larger than *r*, the discrepancy between the two being related to the size of the departure from linearity. *Eta has no sign.* An inspection of the scatterplot will show that in some cases the relationship between the two variables is at some points positive and at other points negative. Eta then only measures the degree of the relationship. Eta is also affected by the number of columns and the frequencies within the columns. There should be enough of the latter to give the means of the various columns stability.

TABLE 16.9. Summary of the Major Correlation Coefficients

		Variables	
Coefficient	Symbol	X	Y
Pearson product-moment	r	Continuous	Continuous
Point biserial	r_{pb}	Continuous	True dichotomy
Biserial	r_b	Continuous	Continuous, but forced into a dichotomy
Tetrachoric	r_t	Continuous, but forced into a dichotomy	Continuous, but forced into a dichotomy
Phi or fourfold	ϕ	True dichotomy	True dichotomy
Spearman rank-order	(rho)	Data in ranks or capable of being ranked	
Kendall's coefficient of concordance	W	Used with three or more sets of ranks	
Correlation ratio	(eta)	Continuous	Continuous or discrete; may take on nominal or other types of values

OTHER CORRELATIONAL TECHNIQUES

We shall conclude this chapter with brief discussions of correlation methods which involve more than two variables, the partial r and the multiple correlation coefficient (R).

The Partial r

It frequently happens that the relationship between two variables is influenced by a third variable. For example, suppose that we have the relationship between intelligence test scores and arithmetic grades for a set of students and also the correlation of these same intelligence scores and grades in English. We also have the relationship between the arithmetic grades and the English grades. Now both of these school subjects are related to intelligence test scores, and they seem to be related to each other. Is this last relationship a true one or is it merely the effects of these two variables being related to a common one, intelligence? With the partial correlation coefficient it is possible to control these effects of intelligence or to "partial them out." We might also ask what is the relationship between English grades and arithmetic grades with the effect of intelligence partialed out. A situation like this is referred to as a partial r of the first order based upon three zero-order r's. The general formula for this partial-order r is:

$$r_{12.3} = \frac{r_{12} - (r_{13}r_{23})}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}} \tag{16.15}$$

We would read $r_{12.3}$ as the correlation between variables one and two with the effects of variable three partialed out. Similarly, it is possible to write a comparable equation for $r_{13.2}$ and $r_{23.1}$.

Second-order partials are those in which the relationship between two variables is computed with the effects of two other variables partialed out. Since these are rarely used they will not be discussed here.

For example, suppose that we have the following three variables:

1 = chronological age
2 = weight
3 = scores on an arithmetic test

For several hundred students we computed the correlations among three variables and obtained the following:

$$r_{12} = .80$$
$$r_{23} = .50$$
$$r_{13} = .60$$

From this we see that we have a correlation between weight and arithmetic which with a sample of this size is significant. Suppose that we now investigate the relationship between weight and scores on an arithmetic test with the effects of chronological age partialed out. This is done below.

$$r_{23.1} = \frac{r_{23} - (r_{12})(r_{13})}{\sqrt{(1 - r_{12}^2)(1 - r_{13}^2)}}$$

$$r_{23.1} = \frac{.50 - (.80)(.60)}{\sqrt{(1 - .80^2)(1 - .60^2)}}$$

$$r_{23.1} = \frac{.50 - .48}{\sqrt{(.36)(.64)}}$$

$$r_{23.1} = \frac{.02}{.48}$$

$$r_{23.1} = .04$$

Now we see that with the effects of chronological age removed, there is no significant relationship between weight and scores on an arithmetic test. Since the partial r is a Pearson r it may be treated as such. Tests of significance are similar. The student who takes more statistics will see that there is a technique known as the analysis of covariance which produces more reliable results than are obtained with this method.

Multiple Correlation

In an earlier chapter we learned how to make predictions on one variable from another. For example, we learned how to predict grade point averages on the basis of intelligence test scores. It was suggested then that prediction can be made more accurate if the predictors were increased in number. For example, we can do a better job of predicting freshman grade point averages if we use in addition to the usual intelligence test other predictors such as scores on a reading test, high school rank, scores on a mathematics test, and so forth. Prediction efficiency continues up to the addition of the fourth or fifth predictor, but after that the slight gains in predictive ability are all out of proportion to the amount of time involved in including these additional predictors in the regression equation.

In this book we are going to take up multiple correlation in its simplest form, the relationship between one variable and a combination of two other variables. Suppose that we have the following three coefficients based upon three variables for a large group of university freshmen:

Variable 1 = grade
Variable 2 = scores on the *Ohio State Psychological Examination*
Variable 3 = scores on the *Cooperative Mathematics Test*

$$r_{12} = .50$$
$$r_{13} = .60$$
$$r_{23} = .40$$

We want to compute the multiple correlation coefficient between freshman grades and the combined effects of the two tests. The formula follows:

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - (2r_{12}r_{13}r_{23})}{1 - r_{23}^2}} \tag{16.16}$$

$$R_{1.23} = \sqrt{\frac{.50^2 + .60^2 - (2)(.50)(.60)(.40)}{1 - .40^2}}$$

$$R_{1.23} = \sqrt{\frac{.25 + .36 - .24}{.84}}$$

$$R_{1.23} = \sqrt{\frac{.37}{.84}}$$

$$R_{1.23} = \sqrt{.4405}$$

$$R_{1.23} = .66$$

The multiple R may be used in the same manner as the zero-order r. The standard error of estimate can be computed.

$$s_{1.23} = s_1\sqrt{1 - R^2} \tag{16.17}$$

This standard error of estimate is interpreted in the same manner as before. It is also possible to set up an index of forecasting efficiency on the basis of the equation $R^2 + K^2 = 1$. Computing charts or abacs have been developed from which the multiple R can be readily read when only three variables are concerned; for example (**9**).

Exercises

1. A scale to measure attitudes toward Russia is being constructed. One of the items is "Do you think that the Russian people like their government?" The scale is pre-tested on a sample of 200. The distribution of "Yes" and "No" responses with total score on the scale is:

	Yes	*No*
95–99	1	0
90–94	6	0
85–89	18	1
80–84	22	1
75–79	31	3
70–74	20	5
65–69	18	9
60–64	12	13
55–59	6	10
50–54	4	8
45–49	1	5
40–44	0	3
35–39	1	0
30–34	0	1
25–29	0	1
	140	60

(a) Compute the point-biserial r between the item and the total score.
(b) Test the r_{pb} for significance.

2. Using the data in Exercise 1 compute the biserial r using formula 16.3.
3. Compute the phi coefficient for the following data:

	Right	*Wrong*
Upper	65	35
Lower	25	75

Is this a significant phi at the 5 percent level?

4. What is the tetrachoric r for the data in Exercise 3?
5. Seven instructors are rated by freshmen and sophomore students on "clarity of presentation." The results are tabulated in this manner:

Instructor	*Freshmen*	*Sophomores*
1	44	58
2	39	42
3	36	18
4	35	22
5	33	31
6	29	38
7	22	38

What is the Spearman rho for these data?

6. Four judges (parole board members) rank eight convicts on "parole readiness."

	Judges			
Convict	*1*	*2*	*3*	*4*
1	1	1	1	1
2	2	4	3	2
3	3	3	2	4
4	4	2	4	3
5	5	6	5	5
6	6	5	6	7
7	7	7	8	6
8	8	8	7	8

(a) Using the coefficient of concordance, indicate the degree of consistency of the judges?
(b) Is the relationship statistically significant at the 1 percent level?

7. Compute η_{xy} for the data in Table 16.4. Is this coefficient significant at the 5 percent level?
8. An English teacher gives a final objective test and a final theme in his course. These do not enter into the course grade. The three variables correlate as follows:

	Theme	*Test*	*Grade*
Theme	*xxx*	.63	.42
Test		*xxx*	.39
Grade			*xxx*

How well does the theme correlate with grades when the effect of whatever the test measures is partialed out?

9. What is the multiple R between grades and the combined effects of the theme and test as given in Exercise 8?
10. Using the data in Exercise 8, p. 93 of Chapter 8, calculate the Spearman rank-order correlation coefficients among the various tests.

17

RELIABILITY, VALIDITY, AND ITEM ANALYSIS

One of the major uses of correlation coefficients is in the computation of reliability, validity, and item statistics. In this chapter we shall first consider a bit of measurement theory, then the various types of reliability coefficients, the standard error of measurement, the statistical aspects of validity, and finally a brief survey of item analysis techniques.

MEASUREMENT THEORY

In modern educational and psychological measurement, each individual score or measurement is considered as being made up of two parts, a true score component and an error score component. This can be expressed as:

$$X = X_t + X_e \tag{17.1}$$

where:

X = any raw score or unit of measurement.

X_t = true score component.

X_e = error score component.

The error component, or error score as it is sometimes called, is held to be random. Suppose that you were to weigh a number of objects. At times there would be slight errors in the way that you read the scale. These deviations would not always be in the same direction. Sometimes your readings would be too high and sometimes they would be too low. In the long run, they would tend to cancel out, and the mean of all of these errors of measurement would be zero. Such variations as these are described as *random* errors and their mean is always zero. But suppose that something was operating in our method or in our apparatus which caused us to always have an error which was always a certain amount on the high side. Errors such as these are described as *systematic*. Their mean would not be zero.

Whenever we give a test, there are many factors which enter into the error component of an individual's score. Some of these are guessing, misreading of an item, daily fluctuations in an individual's health and emotional status, and many physical factors. The size of this error component is related to the reliability of any measuring device. The smaller the error component or error score, the more reliable the measuring instrument. In its simplest form, reliability means consistency. A reliable instrument leads to measurement units which are fairly similar from time to time.

Since any score can be broken into two parts, a true component and an error component, the variance of a test can also be so treated. We can write this as follows:

$$s^2 = s_t^2 + s_e^2 \tag{17.2}$$

which reads that the variance of a test is made up of the variance associated with true scores (true variance) and the variance associated with error scores (error variance).

Next we shall divide equation 17.2 through by a constant, s^2.

$$\frac{s^2}{s^2} = \frac{s_t^2}{s^2} + \frac{s_e^2}{s^2} = 1 \tag{17.3}$$

and by transposing

$$\frac{s_t^2}{s^2} = 1 - \frac{s_e^2}{s^2}$$

In modern test theory reliability is defined as that part of the variance which is true variance, viz,

$$r_{tt} = \frac{s_t^2}{s^2} \tag{17.4}$$

or by substituting an identity for the right-hand part of equation 17.4, we have

$$r_{tt} = 1 - \frac{s_e^2}{s^2} \tag{17.5}$$

which reads that the reliability is 1 minus that part of the total variance which is error variance.

Methods of Computing Reliability Coefficients

There are at the present time a number of techniques used in the computation of reliability coefficients. However, in this book we shall limit ourselves to a consideration of the more basic and simpler ways which are the methods of test-retest, parallel forms, split-halves, and Kuder-Richardson Formula No. 20.

Test-Retest Method. With this technique, a test is administered and then, at a later date, the same test is readministered to the same individuals. A Pearson product-moment correlation coefficient is computed between the two sets of scores. There are various conditions which affect this technique limiting its effectiveness. The length of the time between the two administrations of the test is important in determining the size of the reliability coefficient. In general, the longer the time between the two administrations of the test, the lower the correlation. The research literature contains many studies involving intelligence tests, tests of special abilities, and interest inventories which bear this out. If the period between the administrations is very short, such as the second test immediately following the first, individuals may remember their answers to the items from the first time and put the same responses down the second time without making any effort to react to the test item. Such behavior as this tends to make reliability coefficients too high. When the period between testings is short, memory on many tests is an important factor affecting the results. As the period increases in length such factors as learning, maturation, senescence, and many other variables enter into the situation to lower the correlation coefficient. At the present time this method is infrequently used when calculating the reliability of paper-and-pencil tests. Coefficients computed by this method are frequently called *co-efficients of stability*.

Parallel Form Method. This technique is also referred to as the method of equivalent forms. Using this technique, we administer Form A to a group of individuals and follow this immediately or fairly soon with Form B of the same test. These two forms of the same test are said to be parallel or equivalent because they are made up of the same types of items, covering the same materials; they have the same means and variances; and, if one form correlates to a certain extent with some other measure, then the other form correlates to the same degree.

As in the previously discussed method, a Pearson product-moment correlation coefficient is computed between the two sets of scores. This coefficient is sometimes referred to as a *coefficient of equivalence*. This method is widely used at the present time. It may be called the technique with the most universal applicability. When other methods cannot be used, we can always fall back upon this one. This is especially true when we wish to determine the reliability of speed tests. The memory factor which was important with the test-retest method is ruled out here. But such factors as learning, growth, and change are still present, and here again, the longer the period between the two test administrations, the lower the reliability coefficients tend to be.

The Split-Half Method. An advantage of this method is that only one test is needed for the computation of the reliability coefficient. The test papers are so scored that from every single paper we have two scores. This is usually done by counting the number of odd-numbered items answered correctly and the number of even-numbered items answered correctly. Sometimes other splits are made, such as items 1 and 2 go into the first score, 3 and 4 into the second score, 5 and 6 into the first score, etc. Almost any split will be acceptable except taking the first half of the items against the second half. Because tests are usually made with the easier items first and because students tend to complete the first half of a test and not the second, a division of this sort will result in two tests of a different nature.

Each paper now has an even and an odd score on it, or two other types of scores depending on the type of split made. Again a Pearson product-moment correlation coefficient is computed between the two sets of scores. The name that is often given to a reliability coefficient of this type is *coefficient of internal consistency*.

It so happens that the reliability of a test is directly related to the length of the test. When we scored our test on an odd-even basis, what we actually did was to cut the length of our original test in half. The reliability coefficient which we have computed is then the equivalent of one of a test of half of the size of our original test. We make a correction at this point by using what is known as the Spearman-Brown formula which is as follows:

$$r_{tt} = \frac{2r_{oe}}{1 + r_{oe}} \tag{17.6}$$

where

r_{tt} = the reliability of the original test.

r_{oe} = the reliability coefficient obtained by correlating the scores on the odd items with the scores of the even items.

Suppose that we have a 100-item test. As a result of running a correlation coefficient on the odd scores and the even scores, we obtain a coefficient of .84 ($r_{oe} = .84$). By substitution, we find

$$r_{rr} = \frac{2(.84)}{1 + .84}$$

$$r_{tt} = \frac{1.68}{1.84}$$

$$r_{tt} = .91$$

This Spearman-Brown formula is frequently referred to as the Spearman-Brown prophecy formula as it is used to predict reliability coefficients. For example, suppose we have a 20-item test and we know its reliability. By the use of this formula we can predict what the reliability of this test would be if 60 additional similar and well-made items were added. Or suppose that we have a 100-item test of known reliability and we wish, because of time limitations, to reduce it to a 50-item test. Again, by the use of this Spearman-Brown formula we can estimate the reliability of this shortened test. In its general form, the Spearman-Brown prophecy formula is written as follows:

$$r_{tt} = \frac{Nr}{1 + (N - 1)r} \tag{17.7}$$

where

N = the number of times the test is being lengthened or shortened.

r = the reliability of the test which is being lengthened or shortened.

Here is how this formula is used. Suppose that we have a 20-item test with a reliability coefficient of .60. We wish to know what the reliability of this test would be if 80 more similar and well-made items were added to it to make it a 100-item test. N for this problem is 5 as the new test is five times the length of the original one. The solution is

$$r_{tt} = \frac{5(.60)}{1 + (4).60}$$

$$r_{tt} = \frac{3.00}{1 + 2.40}$$

$$r_{tt} = \frac{3.00}{3.40} = .88$$

To illustrate further, suppose that we have a 110-item test, the length of which is reduced to 55-items. The reliability of the original test is .88. What is the reliability of the shortened test? In this case N is .5 as the new test is one-half the length of the original one. The solution follows:

$$r_{tt} = \frac{(.5)(.88)}{1 + (.5 - 1)(.88)}$$

$$r_{tt} = \frac{.44}{1 + (-.5)(.88)}$$

$$r_{tt} = \frac{.44}{1 + (-.44)}$$

$$r_{tt} = \frac{.44}{.56} = .786$$

Results obtained by the use of this formula agree very well with those obtained when tests are changed in length and the coefficient is computed for the new test.

Internal consistency coefficients are very suitable for use in computing the reliability of academic tests. However, they must not be used with tests in which speed is an important factor. A speed test is defined as one in which the items are so easy that, given enough time, there is no reason why any individual cannot respond correctly to all of the items. On such a test, the scores on the odd-numbered items are likely to be very similar to those obtained from the even-numbered ones and hence the resulting correlation coefficient will be exaggerated.

Kuder-Richardson Formula No. 20. This technique also yields a *coefficient of internal consistency* and as such has the limitations of the split-half method. If an item analysis has been made for a test (this will be discussed later in this chapter), the Kuder-Richardson formula is easily applied to the data to obtain the reliability coefficient. One of the outcomes of an item analysis is a difficulty measure for each item on the test. Difficulty is defined as the proportion or percentage of those responding to an item who answered it correctly. The symbol p is used to represent difficulty. An item with $p = .89$ was answered correctly by 89 percent of those who responded to it.

In working a Kuder-Richardson solution we first set up a worksheet like this:

(1)	(2)	(3)	(4)
Item	p	q $(1 - p)$	pq
1	.60	.40	.2400
2	.30	.70	.2100
3	.71	.39	.2769
etc.		$\Sigma =$	

The first column consists of the number of the item. In the second column the difficulty value (p) of the item is recorded from the item analysis work. The third column is labeled q, which always means $1 - p$. In all statistical work, $p + q = 1$. The last column is labeled pq which is the product of columns (2) and (3). This column should be carried to four decimal places. Then this last column is summed.

The formula for Kuder-Richardson No. 20, is

$$r_{tt} = \frac{n}{n-1}\left[1 - \frac{\Sigma pq}{s^2}\right] \qquad (17.8)$$

where

n = number of items on the test.

s^2 = the variance of the test.

pq = the quantity obtained from the worksheet.

The Size of Reliability Coefficients. In general, reliability coefficients of well-made standardized tests tend to be high, .90 or above. There is no hard and fast rule that says that any reliability has to be of a certain size before any test or measuring instrument can be useful. Today we look upon reliability as a relative thing, and there are certain areas of certain techniques where reliability coefficients fall well below this .90 and the techniques are still used and found to be very useful. Rating scales are examples of this.

As noted above, the length of any test influences the size of the reliability coefficient for that instrument. Since reliability coefficients are correlation coefficients, they too are greatly affected by the range of scores in the sample on which the reliability correlation is computed. The more homogeneous the sample, the lower the reliability coefficient.

Standard Error of Measurement

Rather than use correlation coefficients to express the reliability of a test, some prefer to use a statistic called the standard error of measurement. This, unlike the reliability coefficient, is not affected by the range of scores of the sample tested. It tends to be about the same for samples with different variances. This standard error of measurement is the standard deviation of a sample of scores of an individual about his true score. Suppose that it were possible to administer the same test repeatedly to the same individual, without any changes occurring in the individual. These scores would be different. If we took the mean of all of them, we could use this as an estimate of the individual's true score, and the standard deviation of these scores about this true score would be referred to as the standard error of measurement.

It is impossible to readminister the same test over and over again to the same individual; so the best that we can do is estimate the standard error of measurement. This is done by the following formula:

$$s_e = s\sqrt{1 - r_{tt}} \tag{17.9}$$

where

s_e = standard error of measurement.
s = standard deviation of the test.
r_{tt} = reliability of the test.

This equation is very easily derived from equation 17.5 in the following manner:

$$r_{tt} = 1 - \frac{s_e^2}{s^2}$$

Clearing of fractions and multiplying all terms by s^2,

$$r_{tt}s^2 = s^2 - s_e^2$$

Transposing terms,

$$s_e^2 = s^2 - r_{tt}s^2$$

Factoring the right-hand side of the equation,

$$s_e^2 = s^2(1 - r_{tt})$$

Taking the square root of both sides,

$$s_e = s\sqrt{1 - r_{tt}}$$

Interpretation of the Standard Error of Measurement. Suppose that we administer a test with a reliability of .91 to a group. We find that the standard deviation is 11. A student obtains a score of 77 on this test. First we calculate the standard error of measurement

$$s_e = s\sqrt{1 - r_{tt}}$$
$$s_e = 11\sqrt{1 - .91}$$
$$s_e = 11\sqrt{.09}$$
$$s_e = 11(.3)$$
$$s_e = 3.3$$

Since this standard error of measurement is a standard deviation, it is interpreted as such. We can now make the statement that the chances are two out of three that this individual's obtained score of 77 is not more than 3.3 units from his true score. Or we could make other statements, such as that the chances are 95 in 100 that this obtained score is not more than

6.6 units from his true score and finally that the chances are 99 in 100 that this obtained score is not more than 9.9 units from his true score. Each of these probability statements is based upon the different areas cut off when various standard deviation units are measured from the mean. In this case the true score is taken as the mean.

The smaller this standard error of measurement, the more reliable the test and the more confidence that we can place in any score obtained by using the test. An inspection of equation 17.9 shows that if the test were perfectly reliable, that is, the reliability coefficient was equal to 1, the right-hand part of the equation would reduce to 0 and there would be no standard error of measurement. A perfectly reliable instrument would always yield true scores.

VALIDITY

Kinds of Validity. Very simply there are two basic types of validity, *content* and *statistical*. Since content validity has nothing to do with statistics, we can dispose of it very rapidly by stating that a test has content validity when it measures adequately both the materials and the achievement of the objectives of any school course of similar body of content. It is the type of validity associated with achievement tests, and it is assured when the test constructor takes care that his test samples the materials of the course and as many of the objectives of the course as is possible with paper-and-pencil tests.

Statistical Validity. In its basic form statistical validity is the correlation of a set of test scores with an external measure. This external measure is referred to as a criterion (pl. criteria). For example, we validate intelligence tests by obtaining a set of such scores on a group, college freshmen, for example, and then we later obtain the grade point averages that these freshmen made during their first semester of college. A correlation coefficient is then run between our two sets of measurements. In this illustration the grade point averages are the criteria. Other things used as criteria are ratings of performance, ratings of adjustment, units produced in a certain period of time, amount of sales, number of errors made, etc.

These validity coefficients tend to be much lower than reliability coefficients. An examination of the research over the years will show that they tend to fall within the band of .40–.60, with a median value of about .50. A little reflection will show why this is so. If we consider the relationship between intelligence test scores and grades, we know that there is more than intelligence in the obtaining of grades. We are overlooking such factors as motivation and interests of the student, grading practices of the teachers, emotional adjustment, health of the student, time available for study, and many other factors, all of which influence the grades that a student obtains. An examination of the intelligence test scores for any classroom will show that there are one or two individuals who have high measured ability and low achievement or lower ability and high achievement. Cases like these lower the size of the computed correlation coefficients.

When ratings are used as criteria, we find that the ratings tend to be unreliable because of the nature of the trait being rated, lack of knowledge of the ratees by the judges, and various other errors associated with ratings. Again, the validity coefficients tend to be in the middle of the range. In industrial situations where tests of special ability are widely used, the situation is no better. Here the number of units produced in a given period of time is used as the criterion. Many factors affect the number of units turned out by each worker, and we again have to consider such factors as motivation of the workers, speed of the machines, availability of raw materials, and such physical aspects of the environment as noise, light, heat, presence or absence of other workers, and the like.

It should also be emphasized at this point that no test contains validity or reliability. After all, a test is only a piece of paper with symbols printed upon it. Whether this test has validity and reliability depends upon the manner in which it is used. For one situation any test or measuring device may have high reliability and validity; in another situation both may be low. So we might say that a measuring instrument has any number of validity and reliability coefficients, depending upon how and where it is used.

Validity coefficients are used in predicting. Probably the commonest of these prediction situations is that of predicting freshman grades on the basis of intelligence tests. The first time a test is used, grades are later collected, and then the validity coefficient is computed. Then the regression equation for the data is determined (see Chapter 9). With the next group of freshmen, we can take their intelligence test scores and on the basis of our regression equation predict their freshman grade point averages. In actual practice, rather than use a single predictor, several are used. With these freshmen, we might add to the intelligence test scores high school rank, scores on an English test, scores on a mathematics test, and, perhaps, values from interest inventories. When more than one variable is used, we call this multiple prediction. In most cases the use of several variables raises the correlation coefficient and thus leads to better prediction.

If our validity coefficients were perfect, that is, equal to 1, we would be able to predict an individual's exact score on the second variable by the use of the regression equation. However, such is never the case and we thus have to attach a probability statement to our prediction. In this case the statistic used is the *standard error of estimate* which is written as follows:

$$s_{yx} = s_y \sqrt{1 - r_{xy}^2} \qquad (17.10)$$

where

s_{yx} = the standard error of estimate when predicting Y from X.

s_y = the standard deviation of the Y-variable.

r_{xy} = the validity coefficient, the correlation between the predictor and the criterion.

Suppose that for any given situation we find that the standard error of estimate is .4 and that for the same situation we read from the regression equation that when the score on the X-variable (IQ) is 120, the predicted value of Y (grade point average) is 5.4. We would then make a statement like this: The chances are two out of three that the grade point average for this student whose intelligence test score is 120 is between 5.0 and 5.8, his predicted score plus and minus one standard error of estimate. Since the standard error of estimate is also a standard deviation, it is interpreted as such. If we were to add and subtract two standard errors of estimate (2 × .4), we can say that the chances are 95 in 100 that this individual's score falls between 4.6 and 6.2, that is, his predicted score plus and minus .8.

For any correlation situation there are always two standard errors of estimate. The other one here would be the standard error of estimate when predicting X from Y. This is written as follows:

$$s_{xy} = s_x \sqrt{1 - r_{xy}^2} \tag{17.11}$$

where

s_{xy} = the standard error of estimate when predicting X from Y.
s_x = the standard deviation of the X-variable.
r_{xy} = the validity coefficient.

An examination of the formula for the standard error of estimate shows that if r were perfect (equal to 1), the right-hand part of the equation reduces to 0 and there is no standard error of estimate. From this it follows that the higher our validity, the better we can predict. Let us look at the efficiency of these validity coefficients. We can start out by noting the relationship between the coefficient of determination (a name given to r^2) and the coefficient of nondetermination (k^2).

$$r^2 + k^2 = 1 \tag{17.12}$$

then

$$k^2 = 1 - r^2$$

and

$$k = \sqrt{1 - r^2} \tag{17.13}$$

Instead of $r + k$ equaling 1, the sum of their squares equals 1. When equation (17.13) is solved for various values of r, the corresponding values

TABLE 17.1.
Relationship
Between r and k

r	k
1.00	.00
.95	.31
.90	.44
.80	.60
.70	.71
.60	.80
.50	.87
.40	.92
.30	.95
.20	.98
.10	.995
.00	1.00

of k obtained are shown in Table 17.1. From this it will be noted that as r increases, k decreases, but at a much slower rate. When r has increased to .50, k is equal to .87. When r is equal to .70, the value of k is very close to it. Even when the value of r is in the .90's, the k-values are considerable.

This k-statistic is of little value in itself. You may have noticed that it is one of the terms of the formula for the standard error of estimate. Equation 17.10 could be written

$$s_{yx} = s(k)$$

We also use this statistic in making a statement about the prediction efficiency of our validity coefficient. Suppose that we have a situation where the validity coefficient between predictor and predicted variables is .45. First we solve for k:

$$\begin{aligned} k &= \sqrt{1 - r^2} \\ k &= \sqrt{1 - .45^2} \\ k &= \sqrt{1 - .2025} \\ k &= \sqrt{.7975} \\ k &= .89 \end{aligned}$$

Then we compute our index of forecasting efficiency by subtracting the obtained k from 1 and multiplying the remainder by 100:

$$\begin{aligned} E &= (1 - k)100 \qquad (17.14) \\ E &= (1 - .89)100 \\ E &= (.11)100 \\ E &= 11\% \end{aligned}$$

We can interpret this by saying that with a validity coefficient of .45 we can predict 11 percent better than we would have been able to had there been no relationship between the two variables. In other words we have decreased the size of the errors of prediction by 11 percent.

The above may make it appear that tests are not much good in predicting future behavior. But the point to remember is that we are usually not interested in an individual's particular grade point average or score on another test. We are more concerned about whether he will pass or fail as a student, be successful or eliminated as a trainee, etc. Rather than knowing his specific score, we are interested in his location on the other variable. Our tests do an efficient job in placing individuals in this manner.

Attenuation. Since in any given situation both variables are unreliable to a certain extent, any correlation coefficient computed between the two would tend to be lower than the true relationship between the two variables actually is. This lowered r is referred to as attenuated. Formulas have been developed to correct for this attenuation. The most general one is

$$r_c = \frac{r_{xy}}{\sqrt{r_{xx}r_{yy}}} \qquad (17.15)$$

where

r_c = the correlation between X and Y corrected for attenuation.

r_{xx}, r_{yy} = the reliability coefficient of Test X and criterion Y, respectively.

r_{xy} = the computed validity coefficient.

In actual practice, it is best not to use this formula, for the reliability of our tests is at present quite high and it is doubtful if it will ever be considerably improved. We have to use these tests in our daily work and we must be aware that they are not perfect and that any prediction made with them is apt to be inaccurate to a certain extent. However, many of the criterion measures that we use are low in reliability because of inaccuracies inherent in our measurement techniques. Some workers then feel justified in correcting for attenuation in the criterion scores only. Equation 17.15 then becomes

$$r_c = \frac{r_{xy}}{\sqrt{r_{yy}}} \tag{17.16}$$

ITEM ANALYSIS

To determine the merit of any test item, test results have to be subjected to an item analysis. As a result of this item analysis three things are obtained concerning the item: (1) the difficulty of the item; (2) the discrimination index of the item; and (3) the effectiveness of the distractors. The first of these, the difficulty of the item, is the proportion of individuals who answer the item correctly. The discrimination index is a measure of how well the item separates two groups. The purpose of most tests is to "spread out" the individuals taking it. The item which separates good students from poor ones, adjusted individuals from maladjusted ones, or those with artistic ability from those without such ability is said to discriminate. The third of the above results applies only to multiple-choice items. But since most of our standardized tests and many of the teacher-made tests use this type of item almost exclusively, a thorough item analysis is made to see how the incorrect responses in the multiple-choice item are working. We shall now discuss each of these results of an item analysis in detail.

Item Difficulty. As previously defined in this chapter, the difficulty of an item (p) is the proportion of individuals who answer an item correctly. In a power test, that is, a test in which everyone has had a chance to read every item, the calculation of the item difficulty brings up no problems. But when speed is a factor in a test—and many of our tests are of this nature—adjustments have to be made in the calculation of the item difficulty. Basically, item difficulty is the result of dividing the number who answered the item correctly by the number who took the test. On a speed test many people never get to the answers near the end of the test. In that case the number of correct answers should be divided by the number of individuals who reached the item. The usual procedure in determining how far an individual went on a test is to assume that when the test period was over the individual was working on the item following the last one that he marked on his answer sheet.

All items beyond this one are not assumed to have been reached. On the first half of any test such corrections make little difference; but unless something like this is done on the items near the end of the test, both the item difficulty values and the indices of discrimination will be spuriously large. It also follows that as a result of corrections like this the size of the sample upon which the item statistics depend becomes much smaller, and correspondingly the reliability of the item statistics is reduced. Perhaps in constructing tests, even speed tests, it is a good idea to give the examinees enough time to answer all of the items. One way to do this is to attach a group of buffer items at the end of the test to keep those who finish first busy. These items are not scored.

The difficulty index is a very important item statistic because it is very closely related to item discrimination. Suppose that we take an item with a difficulty of 99 percent which appeared on a test that was given to 100 students. This item makes $99 \times 1 = 99$ discriminations. That is, it separates the individual who missed it from the 99 who answered it correctly. An item with a difficulty of 98 percent makes 98×2 or 196 discriminations. In a similar fashion we could continue this reasoning downward until we come to items of 50 percent difficulty. With these we see that the number of discriminations on this test would be $50 \times 50 = 2500$ discriminations, the maximum number possible. It might appear then that the best test would be made up of items of 50 percent difficulty. If such a test were made it would tend to cut the individuals taking it into two groups, those above the median and those below. With most tests, we wish to discriminate throughout the range. We wish to be able to grade students on the conventional letter system or on some other classification scheme using the results of the test. To have a test which discriminates over the entire range, items are selected which range from very easy to very difficult and which average, in the long run, to a difficulty value of 50 percent. From the viewpoint of item difficulty, a well-made test starts with a few very easy items, continues with the items of increasing difficulty, and ends with a few items which only a very few of the examinees answer correctly. There would be more difficulty values clustering about the center than at either extreme, but there would be a balance so that the average item difficulty is 50 percent.

There are times, however, when we do not wish to spread out individuals for grading purposes. On an examination for national science scholarships the testmakers are interested in selecting the top 5 percent of the applicants. To do this would require the construction of a very difficult test with item difficulty values equal to .05. Since chance or guessing may be important in affecting scores on a test like this, the total scores on such a test should be corrected for guessing before the item analysis is made. The usual formula for correcting for guessing is

$$\text{Score} = R - \frac{W}{N - 1} \tag{17.17}$$

where

R = the number of right answers on a student's paper.
W = the number of wrong answers on a student's paper.
N = the number of possible choices in the test item.

For the ordinary true-false item, $N = 2$. For five-response multiple-choice items, $N = 5$. There are other cases in which one may be interested in accepting, say, 70 percent or 30 percent of the applicants. In these cases items for the test should all be of either .70 or .30 difficulty, respectively.

Item Discrimination. There are two general ways of demonstrating item discrimination: (1) a test of the significance of the difference between two proportions and (2) correlational techniques. In the first method the percent or proportion of individuals who answer the item correctly in the high group is tested against the proportion in the low group. If the difference is a significant one, the item is accepted as being one which discriminates. A disadvantage of this technique is that while it selects those items which discriminate, it does not reveal how well each one discriminates.

In much item analysis work the student will find that it is customary to compare the responses in the top 27 percent of the papers with those in the lowest 27 percent of the papers. This goes back to the early item analysis work of T. L. Kelley who demonstrated that when the high group was made up of the top 50 percent, those papers clustering about the median tended to reduce the sharpness of the difference between the two groups. Comparing the upper 40 percent with the lower 40 percent produced more clear-cut results. This difference becomes increasingly sharp as the papers in the high and low groups become more extreme. Also as the number of papers in the two groups decreases, the standard errors increase and the item statistics become more unreliable. Kelley demonstrated that when the responses of the individuals in the upper 27 percent were compared with those in the lower 27 percent, the ratio of the difference between the means of the two groups over the probable error[1] of the difference between the means was at a maximum. Since that time many workers have accepted his findings and used this value of 27 percent in selecting the high and low groups. A major reason for this is that a number of useful computing devices have been made which are based upon high and low groups consisting of the upper and lower 27 percent. That other percentages will work just as well has been suggested by research which compared results based upon groups using 27 percent with those using a series of other percentages.

In the correlational approach to item analysis a correlation coefficient is computed that shows the relationship of the responses to the total test score. In other words, we are investigating how well the item is doing what the test itself is doing. Suppose that we have the responses of 200 individuals to a 56-item test. To begin with we are going to use a method which uses all of the papers. We set up a distribution of scores on the Y-axis of our scatterplot and since the test item is usually scored on a right or wrong basis we have only two categories on the X-axis (see Table 17.2).

Taking the first paper, note the total score and whether or not the item was answered correctly, and tally the item at the appropriate place on the

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[1] Probable error = .6745 standard error. This is an outmoded statistic.

TABLE 17.2. Responses of Two Hundred Students to One Item of a Fifty-Six-Item Test

Test Score	Right	Wrong
51–53	6	1
48–50	18	2
45–47	22	4
42–44	18	7
39–41	28	6
36–38	15	8
33–35	12	12
30–32	8	10
27–29	5	4
24–26		4
21–23	1	4
18–20		3
15–17		1
12–14		1
	133	67

frequency distributions. Then the next paper is taken, the response tallied, and this process is continued for all of the papers. When the tallying is finished, either a point-biserial or biserial *r* may be computed. If the distribution of test scores is dichotomized as the response to the item is, either the tetrachoric-*r* or the phi coefficient can be used. This same process would have to be repeated for the other 55 items on this test.

Admittedly this is a tremendous amount of clerical work and if it were not for a large number of computing devices that have been set up, many would think twice before getting involved in item analysis work. Flanagan was one of the first to produce one of these short cuts. From his chart correlation coefficients are read directly when one enters the chart with the percent answering the item correctly in the upper or lower 27 percent of the papers. Flanagan's chart is reproduced in Figure 16.2 of Chapter 16. Guilford (**6**) has produced a chart for obtaining phi coefficients when one enters it with the percentages in the high and low groups answering the item correctly (see Chapter 16, Figure 16.3). For this chart, the high and low groups need not be based upon the upper and lower 27 percent of the cases. The only requirement is that there be an equal number of individuals in both groups. A short method for estimating the tetrachoric *r* has already been demonstrated (Chapter 16). Computing devices are also available which shorten the work in obtaining both the biserial and point-biserial, but none of these is as simple as the other methods mentioned above.

Each of these different correlation coefficients may be tested for significance by the use of the different techniques mentioned in Chapter 16. Flanagan's *r* may be taken as an estimate of a Pearson *r*, and the tables used for estimating the significance of a Pearson *r* may be used in testing the significance of these. In selecting items to put on a standardized test or on future editions of a teacher's own test, items with the highest discrimination indices should be selected since the items are being put into the new test on the basis of their item difficulty values. It should be mentioned here that when the upper and lower 27 percent of the papers are used, item difficulty values are based upon the number of individuals in these two extreme groups who answer the item correctly rather than upon all of the papers.

The Effectiveness of Distractors. In estimating the effectiveness of the different distractors of the multiple-choice item two procedures may be used. The responses to the high group can be counted for the different distractors of each test item and the same can be done for the low group. Good distractors are those which are selected to a greater extent by the members of the low-scoring group. When reversals are found, that is more individuals in the high group select a particular response than in the low group, the test-maker considers the distractor and, if he still wishes to use the item, he revises the distractor, trying to make it less appealing to the better students. This analysis of distractors also reveals some which no one in either the high or low group used in answering the test item. Such distractors are just so much dead wood cluttering up the test. When such appear, they should be revised in an attempt to try to get them to work.

The other method of studying the distractors is to obtain the mean score of the individuals who respond to each of the distractors as well as to the correct answer. When this technique is used, it is not unusual to find that the mean score for individuals who selected a certain distractor is higher than the mean score for those who selected the correct answer. Here again we have evidence of a distractor lowering the discriminatory power of an item. When this technique of mean scores is used, the good distractors are associated with lower mean scores than the mean of the papers associated with the correct answer. With this technique also certain distractors will appear that are selected by no one.

Item Analysis When the Number of Papers Is Small. The item analysis techniques described above are all associated with large or fairly large samples. In many situations all that one has is about 30 papers. With such samples a crude sort of item analysis can be made by using the upper ten papers versus the lower ten. The percent who answer the item correctly in each group can then be determined and from these an estimate of the item difficulty can be made. Estimates as to whether or not an item is discriminating can be made by an inspection of the number of individuals in the high and low groups for each item. Those items with no or a negative discrimination can be spotted at once. For the category of actually discriminating items, we might adopt such a standard as that there has to be a difference of at least two between the frequency of the high and low group before the item is considered as discriminating. While this technique is crude, it is better than nothing and will be helpful in making better classroom tests.[2]

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[2] Item analysis techniques are discussed in detail in any educational or psychological measurement book (**3**, **12**).

Exercises

1. Below are tabulated the responses of 25 individuals to each of the items on a 15-item test.

Item No.		1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	Score
Individual	A	+	+	+	0	+	+	0	0	0	0	+	0	0	0	0	6
	B	+	+	+	+	+	+	0	+	+	+	0	+	+	+	+	13
	C	0	+	+	+	+	+	0	0	0	+	0	+	+	+	0	9
	D	+	0	+	0	0	+	+	+	0	0	+	0	0	+	0	7
	E	+	+	+	+	+	+	+	+	+	0	+	+	+	+	+	14
	F	+	0	0	+	+	+	+	+	+	+	+	+	+	0	+	12
	G	+	+	+	0	0	+	+	+	0	0	+	+	+	+	+	11
	H	0	0	+	+	0	+	+	0	0	0	0	0	0	0	0	4
	I	+	+	0	+	+	0	+	+	+	+	+	0	+	+	+	12
	J	+	+	+	+	0	+	+	+	+	0	+	+	+	+	+	13
	K	+	0	0	+	+	0	0	+	0	0	+	0	+	0	0	6
	L	0	+	+	+	+	+	0	0	0	0	+	0	+	0	+	8
	M	+	+	+	0	0	+	0	0	0	0	+	0	+	+	+	8
	N	+	+	+	+	+	+	+	+	0	+	+	0	+	+	+	13
	O	+	+	+	+	0	+	0	0	0	+	0	+	+	+	0	9
	P	+	+	+	+	+	0	+	+	+	0	0	0	0	0	0	8
	Q	+	+	+	+	+	+	0	0	0	+	0	0	+	0	0	8
	R	+	+	+	+	+	+	0	+	+	+	0	+	0	0	0	10
	S	0	+	0	0	0	+	0	+	0	0	0	0	0	0	0	3
	T	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+	15
	U	+	+	0	+	+	+	0	+	0	+	0	0	0	+	+	9
	V	0	+	+	+	0	+	0	+	0	0	+	0	0	0	0	6
	W	+	+	0	+	+	+	+	+	+	+	+	0	+	+	+	13
	X	+	+	+	+	+	+	0	+	+	+	+	0	+	+	0	12
	Y	+	+	+	+	0	+	+	+	+	+	+	+	+	+	+	14
No. Correct		20	21	19	20	16	22	12	18	11	13	16	10	17	15	13	243

(a) Obtain an odd and even score for each individual and then compute the reliability of the test.
(b) Using Kuder-Richardson formula 20 compute the reliability of the test.
(c) Calculate the standard deviation of the test scores.
(d) What is the standard error of measurement for this test?
(e) Using an individual with a score of 10 on this test, interpret the standard error of measurement.

2. (a) A certain 25-item test has a reliability coefficient of .72. To this are added 75 similar and well-made items. What is the reliability of the new test?
(b) Suppose that a test made of 150 items takes 80 minutes to administer. The author of the test decides to shorten it to 50 items so that it will be a 25–30 minute test. The reliability of the original test is .93. What might you expect the reliability of the shortened test to be?
(c) Another test has a reliability coefficient of .60. This test is made up of 20 items. How many well-made and similar items will have to be added to increase the reliability to .90?

3. (a) A certain test has a reliability coefficient of .92 and a validity coefficient of .48 with criterion Y. The standard deviation of this criterion is 10.2 and that of the test is 9.6. Calculate the standard error of estimate when pre-predicting *Y* from *X*.

(b) Suppose that from a regression equation for the above data you calculate a criterion score of 77 for an individual. How would you associate the standard error of estimate with this predicted score?

(c) Calculate the index of forecasting efficiency for this validity coefficient. What does this mean?

4. Sometimes we square the validity coefficient and call it a coefficient of determination. We then interpret this as being the amount of variance in Y that is accounted for or determined by the variance in X. Using the same line of reasoning, how might one best describe k^2?
5. Take the data in Exercise 3(a) and apply the attenuation formula for correcting in both the criterion and test scores. Exactly what have you done? Can you defend the use of this statistic?
6. Investigate the pros and cons of the use of correction formulas in scoring test papers. Any educational and psychological measurement book may be consulted.
7. If test papers are available, apply an item analysis, calculating for each item the difficulty value and the discrimination index. Use any of the computing charts or devices discussed in this chapter. Summarize your results. Examine the items that are no good. Have you any ideas why they did not hold up in the item analysis?

18

DISTRIBUTION-FREE STATISTICS

The past decade has seen an increasing interest in a large number of statistics that are referred to as "distribution-free" or "non-parametric." Both these terms refer to the fact that these statistics are not based upon the assumption of a normal sampling distribution. Most of the statistics studied so far with the exception of chi square and several of the correlation coefficients require the assumption that the sampling distributions are normal. Since these statistics are based upon no such assumption, they are more often applicable to small samples and those in which an interval scale is not used in collecting data.

In one of the early articles on the use of distribution-free statistics, Moses (**11**) noted among the advantages of such statistics that (1) they are easy to apply; (2) they apply to rank data; (3) they can be used when two sets of observations come from different populations; (4) there is no other alternative when the size of the sample is really small; and (5) whatever may be the shape of the distribution from which the sample distribution is drawn, the nonparametric test of a specified significance level actually has that significance level. Among the disadvantages, he notes that they are not as efficient as the parametric statistics when used as tests of significance. From this it follows generally that a parametric test should be used if it is possible. In addition, he notes that information on these tests is scattered in the literature. This situation has been changed with the publication of such books as those of Siegel (**13**) and Tate and Clelland (**16**).

In this chapter a few of the simpler distribution-free statistics will be discussed. The student who wishes more information is referred to either of the above books.

TESTS FOR CORRELATED SAMPLES

The Sign Test. Among the simplest of these tests is the *sign* test. In discussing this test we shall refer to the material in Table 18.1. In that table data are found in pairs. Individuals are matched and assigned to one of the two groups at random at the beginning of the experiment. A treatment is applied and the resulting scores for the pairs of individuals are shown in the table. In the use of a test like this it is assumed that the variable under consideration is distributed continuously and that both members of each pair are treated similarly except for the experimental variable.

TABLE 18.1. The Sign Test

X	Y	$X - Y$
16	4	+
12	18	−
22	10	+
16	14	+
14	12	+
10	14	−
20	10	+
18	12	+
10	4	+
22	12	+

In analyzing data using this test, we consider each pair of scores separately, noting whether the second score is smaller or larger than the first, and then we assign the appropriate sign as shown. Ties may be handled by dropping them from the computations or by giving one tie a plus sign and the next a minus sign, and so on.

The null hypothesis tested here is that the median change is 0. This hypothesis is rejected if there are too few of any one sign. Where there are 10 or less pairs we apply a test of significance by the use of the binomial expansion with $p = .5$ and $N =$ the number of pairs. For our data, $N = 10$. From Table 18.1, we note that we have 8 pluses and 2 minuses. By chance we would expect 5 of each. The question is, does this frequency of 8 pluses differ significantly from what we would expect by chance? By the use of Pascal's triangle, Table 10.2, we note that the probability of obtaining 8 or more + signs is equal to $\frac{45 + 10 + 1}{1024}$ which is .0547. Making a one-tailed test, we could reject our hypothesis slightly beyond the 5 percent level. Doubling this probability for a two-tailed test would make us unable to reject the null hypothesis.

When the number of pairs is greater than 10, the results may be evaluated using a $\tilde{z}$-score and the formula for the mean and standard deviation of the binomial (formulas 10.2 and 10.3). A correction for discontinuity must also be made (see description in Chapter 10).

Since this test uses only information on the direction of the differences between pairs, it is not the most efficient test. The next test combines with the sign of the difference the relative size of the difference. It is a more powerful test.

Wilcoxon's Matched-Pairs Signed-Ranks Test. This test is applied to pairs of scores as shown in Table 18.2.

The first step is to obtain the difference for each of the pairs of scores. Then we rank the absolute values of these differences. This means to disregard the signs. In this problem the difference of 3 is the smallest and is given the rank of 1. Scores of 8 and −8 are tied for ranks 3 and 4; hence they are given ranks of 3.5. In previous ranking problems the student will recall that the highest score was given a rank of 1 (cf. the computation of ranks for the Spearman rank-order correlation method in Chapter 16). Pairs which have no differences are disregarded in the computations. In the last column the ranks with the less frequent sign are recorded and summed. This sum is given the symbol T. For data like these we can obtain the mean sum of the ranks by the following formula:

TABLE 18.2. Illustration of the Wilcoxon Matched-Pairs Signed-Ranks Test

X_e	X_c	Difference	Absolute Rank of Difference	Ranks with Less Frequent Sign
72	68	4	2	
70	89	−19	7	7
64	72	−8	3.5	3.5
54	68	−14	5	5
72	69	3	1	
64	48	16	6	
76	68	8	3.5	
				$\Sigma = 15.5 = T$

$$\bar{T} = \frac{N(N+1)}{4} \tag{18.1}$$

where

$\bar{T}$ = mean sum of the ranks.
N = Number of ranks.

For these data in Table 18.2, $\bar{T} = \frac{(7)(8)}{4} = 14$. If there is no difference between the two groups, T should be the same as $\bar{T}$. If we can show that T differs significantly from $\bar{T}$, we can reject a null hypothesis of no difference. For smaller sums of ranks, for N between 7 and 25, Wilcoxon has tables (Table IX in Appendix) which may be entered to test the significance of the deviation $\bar{T} - T$. For our data this difference is 1.5 and with $N = 7$ we find that this difference is not significant.

When N is greater than 25, the sum of the ranks may be taken as normally distributed with

$$\bar{T} = \frac{(N)(N+1)}{4} \tag{18.2}$$

$$s = \sqrt{\frac{(2N+1)\bar{T}}{6}} \text{ or } \sqrt{\frac{(N)(N+1)(2N+1)}{24}} \tag{18.3}$$

and the sum of the T's may be treated with the familiar z-test with $T - \bar{T}$ as the deviation.

TESTS FOR UNCORRELATED DATA

The Median Test. This test is applied to see if two groups come from populations which have the same median. In using this test the size of the two samples need not be the same. In Table 18.3 are shown arithmetic addition scores for 27 individuals. We shall test the hypothesis of no difference between these two sets of scores by using the median test. Group X contains 13 scores and Group Y, 14. We first compute the median of the entire set of 27 scores. For these data we find this to be 11.

TABLE 18.3. Scores on an Addition Test

X	Y
12	7
16	12
18	14
7	18
6	5
4	16
11	9
12	10
8	14
20	3
18	18
16	9
10	7
	4

If both of these distributions came from the same or from similar populations, half of the X-values and half of the Y-values would lie above this median of 11, and half of each of the two distributions would lie below it. A contingency table is set up as follows:

	X	Y	
Above Median	7 a	6 b	13 k
Not Above Median	6 c	8 d	14 l
	13 m	14 n	27 N

We next make a test of significance using chi square as follows:

$$\chi^2 = \frac{N\left[(ad - bc) - \frac{N}{2}\right]^2}{(k)(l)(m)(n)}$$

$$\chi^2 = \frac{27\left[(56 - 36) - \frac{27}{2}\right]^2}{(13)(14)(13)(14)}$$

$$\chi^2 = \frac{27(42.25)}{33124}$$

$$\chi^2 = .034$$

which with 1 *df* is not significant.

When the cell frequencies in the contingency table become very small, 1 or 2, this chi square solution, even when corrected by Yates' correction for continuity, should not be used. A more complicated method, known as Fisher's exact method, must be used. This may be found in McNemar (**10**, pp. 240–242) or Walker and Lev (**18**, pp. 103–106).

The median test is also applicable when there are more than two groups. The technique is the same. First the median of all of the scores taken together is obtained. Then a contingency table similar to the one above is set up containing the number of scores above the median and the number not above the median for each of the groups. Chi square is then applied to the data in the contingency table and the appropriate conclusion drawn. For this type of problem the expected frequency for a cell is half of the number of cases in the group. It should be noted that when a score falls exactly at the median, this score is included in the group referred to as "not above the median."

Mann-Whitney U-Test. A more powerful test than the median test for uncorrelated data is the Mann-Whitney *U*-test. This test is used with independently drawn random samples, and the size of the samples need not be the same. This test is actually an extension of Wilcoxon's composite-rank method, *T*-test, which was developed to handle independent samples of equal size. A simple application of this method is shown in Table 18.4. In this table we have two samples of 7 each. Suppose one to be the control group and the other the experimental group. The null hypothesis that we are testing is that the two groups came from the same population.

TABLE 18.4. Two Small Independent Samples Illustrating the Use of the Mann-Whitney *U*-Test

X	Y	R_x	R_y
16	8	11	3.5
18	19	12	13
14	6	10	1
8	9	3.5	5
13	10	9	6.5
10	7	6.5	2
12	20	8	14
		$\Sigma R_x = 60.0$	$\Sigma R_y = 45.0 = U$

The first and basic part of this technique is to rank all of the scores in one composite distribution in an increasing order of size, algebraic signs being considered when negative scores are present. In distribution *Y*, the score of 6 is the lowest one; so it receives a rank of 1. The score of 7 receives a rank of 2. The score of 8 in the distribution of *X* and the score of 8 in the *Y*-distribution are tied for ranks 3 and 4; hence they receive an average rank of 3.5. This ranking is continued until all scores receive a rank. Then the two columns of ranks are summed.

At this point an arithmetic check can be made because the sum of the two columns of ranks must be equal to $\frac{N(N+1)}{2}$. For these data

$$\Sigma R_x + \Sigma R_y = \frac{N(N+1)}{2}$$

$$60 + 45 = \frac{14\,(14+1)}{2}$$

$$105 = \frac{210}{2} = 105$$

If there is no difference between the two sets of scores, the sum of the ranks for the X-variable will equal the sum of the ranks for Y. The hypothesis of no difference is rejected if the discrepancy or difference is too large to be accounted for by chance alone. For data in which the number in the variables is between 2 and 20, a table is available that shows the size of the smaller sum of ranks needed to reject the null hypothesis at various levels of significance. For the data in Table 18.4, N_1 (the frequency of the X-variable) is 7, as is also N_2, the number in the Y-category. From Table X in the Appendix we find that for these values of N_1 and N_2 the smaller sum of the two ranks (U) may not exceed 36 for a significant difference at the 5 percent level to occur. Our smallest sum of ranks is 45. Our null hypothesis stands for these data.

When either N_1 or N_2 is larger than 20, we solve for the U-statistic which is considered to be normally distributed for samples of this size. To illustrate this technique, some fictitious data have been assembled in Table 18.5. In

TABLE 18.5. Illustration of Mann-Whitney *U*-Test for Larger Sets of Data

X	Y	R_X	R_Y
7	18	12.5	31
10	14	18	27
12	15	21.5	29
6	16	10	30
8	12	14.5	21.5
4	10	6.5	18
6	8	10	14.5
14	13	27	24
2	13	2.5	24
4	3	6.5	4.5
6		10	
7		12.5	
3		4.5	
5		8	
2		2.5	
1		1	
13		24	
14		27	
11		20	
10		18	
9		16	
$N_1 = 21$	$N_2 = 10$	$\Sigma R_X = 272.5$	$\Sigma R_Y = 223.5$

Group X (N_1) there are 21 cases and in Group Y (N_2) there are 10. The data are treated exactly as they were previously. First the two sets of scores are ranked together, with the smallest score being given the smallest rank. Then these two columns of ranks are summed. Again these can be checked. $\frac{(N)(N+1)}{2}$ = sum of the two sets of ranks. Next U_1 is found by the following formula:

$$U_1 = N_1N_2 + \frac{N_1(N_1+1)}{2} - \Sigma R_x \tag{18.4}$$

$$U_1 = (21)(10) + \frac{21(21+1)}{2} - 272.5$$

$$U_1 = 210 + 231 - 272.5$$

$$U_1 = 168.5$$

The $\tilde{z}$-ratio is then computed by the use of the following equation:

$$\tilde{z} = \frac{U_1 - \frac{N_1N_2}{2}}{\sqrt{\frac{N_1N_2(N_1+N_2+1)}{12}}} \tag{18.5}$$

where the numerator is U_1 minus the second term which is the mean of U and the denominator is the standard deviation of the U.

$$\tilde{z} = \frac{168.5 - \frac{(21)(10)}{2}}{\sqrt{\frac{(21)(10)(21+10+1)}{12}}}$$

$$\tilde{z} = \frac{168.5 - 105}{\sqrt{6720/12}}$$

$$\tilde{z} = \frac{63.5}{\sqrt{560}}$$

$$\tilde{z} = \frac{63.5}{23.7}$$

$$\tilde{z} = 2.68$$

A z-value of this size leads to the rejection of the null hypothesis at the 1 percent level.

This problem could have also been solved for U_2.

$$U_2 = N_1N_2 + \frac{N_2(N_2+1)}{2} - \Sigma R_y$$

$$U_2 = (21)(10) + \frac{10\,(10+1)}{2} - 223.5$$

$$U_2 = 210 + 55 - 223.5$$

$$U_2 = 41.5$$

If this value of 41.5 is placed in the numerator of formula 17.5, it will be found that the value of the numerator of $\tilde{z}$ this time is -63.5. It does not matter, then, for which U-value we solve. The size of the $\tilde{z}$ is unaffected, but the sign is affected. In a two-tailed test this makes no difference in the interpretation of z.

The Mann-Whitney test is said by Siegel to be a powerful test and an excellent substitute for the t-test **(13)**.

Wald-Wolfowitz Runs Test. Another test applicable to unrelated samples is the Wald-Wolfowitz runs test. With this technique we test the hypothesis of no difference in respect to average value, variability, skewness, etc. In Table 18.6 are two sets of data taken on two independent samples. The scores are arranged in a continuous series from high to low and the identity of the distribution from which the score came is recorded below it. The data in Table 18.6 are set up as follows:

TABLE 18.6. Two Sets of Independent Data for the Wald-Wolfowitz Runs Test

X	Y
12	7
8	9
6	16
18	17
14	20
15	19
3	2
5	10
4	22
1	21
$N_1 = 10$	$N_2 = 10$

$$\underbrace{\overset{22}{Y}\ \overset{21}{Y}\ \overset{20}{Y}\ \overset{19}{Y}}_{1}\ \ \underbrace{\overset{18}{X}}_{2}\ \ \underbrace{\overset{17}{Y}\ \overset{16}{Y}}_{3}\ \ \underbrace{\overset{15}{X}\ \overset{14}{X}\ \overset{12}{X}}_{4}\ \ \underbrace{\overset{10}{Y}\ \overset{9}{Y}}_{5}\ \ \underbrace{\overset{8}{X}}_{6}\ \ \underbrace{\overset{7}{Y}}_{7}$$

$$\underbrace{\overset{6}{X}\ \overset{5}{X}\ \overset{4}{X}\ \overset{3}{X}}_{8}\ \ \underbrace{\overset{2}{Y}}_{9}\ \ \underbrace{\overset{1}{X}}_{10}$$

After this is done the numbers of runs is determined. Starting with the score of 22, we have four scores in a row that came from the Y-distribution. These four scores constitute the first run. The score of 18 alone is from the X-distribution. This is run 2. All the runs are then marked as shown and the number of runs counted.

If the two sets of data are from the same population, we would expect the number of runs to be large. That is, there would be a good mixture of high, medium, and low scores in each distribution and this would lead to a large number of short runs. When N_1 and N_2 (the numbers in each distribution) are 20 or less, the significance of the number of runs observed can be tested directly from tables (Table XI in the Appendix). From these tables we see that for the values of N in our sample the number of runs is not significant and, hence, there is no difference between the two samples.

When the size of the two samples is over 20, the significance of the difference is tested as follows:

$$\tilde{z} = \frac{\left(r - \dfrac{2N_1N_2}{N_1 + N_2} + 1)\right) - .5}{\dfrac{2N_1N_2(2N_1N_2 - N_1 - N_2)}{(N_1 + N_2)^2(N_1 + N_2 - 1)}} \tag{18.6}$$

where

r = number of runs.

$\dfrac{2N_1N_2}{N_1 + N_2}$ = mean of the runs.

Denominator = the standard deviation.

$-.5$ corrects the term for continuity.

With this technique the problem of ties is an important one. If the ties are all in one sample or the other, no problem arises. If the ties are in both samples, they should be mixed in all possible ways. If the results agree by all three ways, there is no problem. But if the results differ, the probability of each z is determined and the average of these taken in the evaluation of the null hypothesis. As an example, suppose that in two series of scores, a score of 14 appears three times, twice in the X-distribution and once in the Y-distribution. It could be arranged in the series of scores three ways:

$$\begin{matrix} 14 & 14 & 14 \\ X & X & Y \end{matrix} \text{ or } \begin{matrix} 14 & 14 & 14 \\ X & Y & X \end{matrix} \text{ or } \begin{matrix} 14 & 14 & 14 \\ Y & X & X \end{matrix}$$

Then for this series of ties, the problem would have to be solved three times, determining the different number of runs which the various combinations cause.

This test is not as good as some of the others because of the general nature of the hypothesis that may be tested with it. Moses (**11**) states that tests which test hypotheses against many alternatives simultaneously are not very good in the prevention of the acceptance of the null hypothesis erroneously with respect to one alternative. Since this test is designed to disclose differences of many types, it is not as efficient as the Mann-Whitney test or others which are set up to test specific differences.

Kruskal-Wallis H-Test. The Kruskal-Wallis H-test is an extension of the Wilcoxon matched-pairs rank-test for more than two samples. It tests whether a group of independent samples are from the same or different populations. Data to illustrate this test are presented in Table 18.7.

Table 18.7 shows three sets of data. The scores of the three sets are combined and ranked with the lowest score receiving a rank value of 1. Ties are treated in the usual fashion for ranking data. Then the ranks of

TABLE 18.7. The Kruskal-Wallis Test

X	R_x	Y	R_Y	Z	R_z
12	13.5	13	16	13	16
16	21	18	22	14	19
14	19	14	19	7	8.5
2	1.5	13	16	8	10.5
12	13.5	8	10.5	4	4.5
		7	8.5	3	3
		6	7	2	1.5
		4	4.5	5	6
				9	12
	$\Sigma R_x = 68.5$		$\Sigma R_Y = 103.5$		$\Sigma R_z = 81.0$

the three columns are summed. If the sample proportions are similar, the total sum of the ranks would be divided proportionately among the various samples on the basis of sample size. The sums of the ranks of the various samples are then tested against these proportions of the total sum of ranks, by use of the following formula:

$$H = \frac{12}{N(N+1)}\left[\sum \frac{R_i^2}{N_i}\right] - 3(N+1) \tag{18.7}$$

where

N = number in all samples combined.

R_i = sum of ranks and N_i the numbers in i samples.

For the data in Table 18.7

$$H = \frac{12}{(22)(23)}\left[\frac{(68.5)^2}{5} + \frac{(103.5)^2}{8} + \frac{(81)^2}{9}\right] - 3(23)$$

$$= \frac{12}{506}[938.45 + 1339.03 + 729] - 69$$

$$= .0237(3006.48) - 69$$

$$= 71.3 - 69$$

$$= 2.3$$

If the number of cases in the samples is from one to five, special tables are used in the interpretation of H (Table XII in the Appendix). When the samples contain five or more cases, H is interpreted as chi square with the number of samples minus one degree of freedom. For this problem $df = 2$ and this H is not significant.

This is a powerful nonparametric test.

Exercises

1. In the data below, column X represents 12 scores of members of a control group in an experiment. Column Y represents the scores of 12 matched

individuals who were given the same test after a period of stress. Using the *sign test*, test the hypothesis of no difference.

X	*Y*
46	36
68	50
60	58
58	40
42	44
43	43
40	29
56	36
38	46
58	48
42	38
48	42

2. Using the data in Exercise 1, test the hypothesis of no difference using Wilcoxon's matched-pairs signed-ranks test.
3. Below are the scores of a group of normals and a group of psychotics on the *Picture Completion Scale of the Wechsler-Bellevue*. Test the hypothesis of no difference using the Median Test.

N	*P*
6	7
6	2
14	12
13	13
15	8
6	6
8	4
7	2
10	2
14	12
10	
14	

4. In the data below are three samples of scores obtained on the *Arithmetic Scale of the Wechsler-Bellevue*. Apply the median test to all three samples.

A	*B*	*C*
8	2	12
7	4	3
14	6	10
10	14	4
8	10	14
6	8	11
	6	10
	2	8

5. Apply the Mann-Whitney U-test to the data in Exercise 3.
6. Using the data in columns B and C of Exercise 4, apply the Mann-Whitney U-test.

7. Two groups of children made the following scores on the vocabulary test of the *Stanford-Binet*. By use of the Mann-Whitney U-test, test for differences in the two groups.

Group I		*Group II*
10	33	14
18	36	18
36	26	22
22	24	16
28	31	38
29	13	26
32	19	28
15	23	12
18	25	
36	27	11
21	32	19
27		16

8. Apply the Wald-Wolfowitz runs test to the data in Exercise 7 above.

9. Using the data in Problem 4, test the hypothesis of no difference using the Kruskal-Wallis H-test.

10. The following data from Borislow[1] show the changed response scores of three groups on the *Edwards Personal Preference Schedule.*

Control Group	*Social Desirability Group*	*Personal Desirability Group*
47	76	87
38	76	76
34	75	52
32	73	51
32	62	50
30	47	42
		38

Test the differences among these three groups, using any and all appropriate nonparametric tests.

---

[1] *Journal of Applied Psychology*, 1958, **42**, 25.

FORMULAS

Number		*Page*
4.1	$\bar{X} = \frac{\Sigma X}{N}$	29
4.2	$\bar{X} = M' + \frac{\Sigma f x^1}{N}(i)$	30
5.1	$Q = \frac{Q_3 - Q_1}{2}$	41
5.2	$x = X - \bar{X}$	43
5.3	$AD = \frac{\Sigma \lvert x \rvert}{N}$	43
5.4	$s = \sqrt{\frac{\Sigma x^2}{N - 1}}$	45
5.5	$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{N}$	46
5.6	$\Sigma x^2 = i^2 \left[\Sigma f x'^2 - \frac{(\Sigma f x')^2}{N} \right]$	47
5.7	Charlier's checks $\Sigma f(x + 1)^2 = \Sigma f x'^2 + 2\Sigma f x' + \Sigma f$	47
5.8	$s^2 = \frac{\Sigma x^2}{N - 1}$	49
5.9	Sheppard's correction $s_c = \sqrt{s^2 - \frac{i^2}{12}}$	50
5.10	Averaging standard deviations $s_T = \sqrt{\frac{N_A(\bar{X}_A{}^2 + s_A{}^2) + N_B(\bar{X}_B{}^2 + s_B{}^2)}{N_A + N_B} - \bar{X}_T{}^2}$	50
6.1	$z = \frac{X - \bar{X}}{s} = \frac{x}{s}$	60

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7.1 Equation for the normal curve 68

$$y = \frac{N}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

8.1 Pearson *r*, standard score formula 83

$$r = \frac{\Sigma z_x z_y}{N}$$

8.2 Pearson *r*, deviation formula 84

$$r = \frac{\Sigma xy}{Ns_x s_y}$$

8.3 Pearson *r*, deviation formula (alternate) 84

$$r = \frac{\Sigma xy}{\sqrt{(\Sigma x^2)(\Sigma y^2)}}$$

8.4 Pearson *r*, raw score or machine formula 85

$$r = \frac{N\Sigma XY - (\Sigma X)(\Sigma Y)}{\sqrt{[N\Sigma X^2 - (\Sigma X)^2][N\Sigma Y^2 - (\Sigma Y)^2]}}$$

8.5 Pearson *r*, scattergram formula 87

$$r = \frac{\Sigma x'y' - \frac{(\Sigma fx')(\Sigma fy')}{N}}{\sqrt{\left[\Sigma fx'^2 - \frac{(\Sigma fx')^2}{N}\right]\left[\Sigma fy'^2 - \frac{(\Sigma fy')^2}{N}\right]}}$$

9.1 Equation of the straight line 94

$$Y = a + bX$$

9.2 Equation of the straight line used in regression analysis 96

$$Y' = a + bX$$

9.3 *b*-coefficient for predicting *Y* from *X* 96

$$b_{yx} = \frac{\Sigma XY - \frac{(\Sigma X)(\Sigma Y)}{N}}{\Sigma X^2 - \frac{(\Sigma X)^2}{N}}$$

9.4 *b*-coefficient for predicting *Y* from *X* in deviation form 97

$$b_{yx} = \frac{\Sigma xy}{\Sigma x^2}$$

9.5 *a*-coefficient used in predicting *Y* from *X* 97

$$a_{yx} = \bar{Y} - \bar{X}(b_{yx})$$

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9.6 *b*-coefficient for predicting X from Y in deviation form 97

$$b_{xy} = \frac{\Sigma xy}{\Sigma y^2}$$

9.7 *a*-coefficient used in predicting X from Y 97

$$a_{xy} = \bar{X} - \bar{Y}(b_{xy})$$

9.8 Standard error of estimate, raw score form 100

$$s_{yx} = \sqrt{\frac{\Sigma(Y - Y')^2}{N - 2}}$$

9.9 Standard error of estimate, deviation formula 100

$$s_{yx} = \sqrt{\frac{\Sigma y^2 - \frac{(\Sigma xy)^2}{\Sigma x^2}}{N - 2}}$$

9.10 Standard error of estimate, in terms of test statistics 101

$$s_{yx} = s_y\sqrt{1 - r_{xy}{}^2}$$

10.1 The binomial expansion 105

$$(p + q)^n = p^n + \frac{n}{1}p^{(n-1)}q + \frac{n(n - 1)}{(1)(2)}p^{(n-2)}q^2 + \frac{n(n - 1)(n - 2)}{(1)(2)(3)}p^{(n-3)}q^3 + \ldots + q^n$$

10.2 Mean of the binomial 109

$$m = np$$

10.3 Standard deviation of the binomial 109

$$\sigma = \sqrt{npq}$$

11.1 Standard error of the mean 117

$$s_{\bar{X}} = \frac{s}{\sqrt{N}}$$

11.2 5 percent confidence interval, large N, mean 120

$$\bar{X} \pm (1.96)\,(s_{\bar{X}})$$

11.3 1 percent confidence interval, large N, mean 120

$$\bar{X} \pm (2.58)\,(s_{\bar{X}})$$

11.4 Standard error of the median 121

$$s_{\text{Mdn}} = \frac{1.253s}{\sqrt{N}}$$

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11.5 Standard error of a proportion 121

$$s_p = \sqrt{\frac{pq}{N}}$$

11.6 Standard error of a percentage in terms of proportions 122

$$s_P = 100\sqrt{\frac{pq}{N}}$$

11.7 Standard error of a percentage in percents 122

$$s_P = \sqrt{\frac{PQ}{N}}$$

12.1 Standard error of the difference between two means, uncorrelated data 124

$$s_{D_{\bar{x}}} = \sqrt{s_{\bar{x}_1}{}^2 + s_{\bar{x}_2}{}^2}$$

12.2 Test of the difference between two means, $\tilde{z}$-test 124

$$\tilde{z} = \frac{D_{\bar{x}}}{s_{D_{\bar{x}}}}$$

12.3 Standard error of the difference between two means, correlated data 128

$$s_{D_{\bar{x}}} = \sqrt{s_{\bar{x}_1}{}^2 + s_{\bar{x}_2}{}^2 - 2(r)(s_{\bar{x}_1})(s_{\bar{x}_2})}$$

12.4 *F*-test 134

$$F = \frac{s_1{}^2}{s_2{}^2}$$

12.5 Standard error of the difference between two means, small sample, uncorrelated data 134

$$s_{D_{\bar{x}}} = \sqrt{\frac{\Sigma x_1{}^2 + \Sigma x_2{}^2}{(N_1 + N_2) - 2}\left(\frac{1}{N_1} + \frac{1}{N_2}\right)}$$

12.6 Testing the significance of changes 136

$$\tilde{z} = \frac{D_E D_C}{\sqrt{s_{D_E}{}^2 + s_{D_C}{}^2}}$$

12.7 Standard error of a proportion 137

$$s_p = \sqrt{\frac{pq}{N}}$$

12.8 $\tilde{z}$-test for the difference between two proportions, uncorrelated data 137

$$\tilde{z} = \frac{p_1 - p_2}{\sqrt{s_{p_1}{}^2 + s_{p_2}{}^2}}$$

Number		Page
12.9	Standard error of the difference between two proportions, uncorrelated data $s_{D_p} = \sqrt{s_{p_1}^2 + s_{p_2}^2}$	137
12.10	Alternate formula for the standard error of the difference between two proportions, uncorrelated data $s_{D_p} = \sqrt{\frac{p_1q_1}{N_1} + \frac{p_2q_2}{N_2}}$	137
12.11	Standard error of the difference between two proportions, small samples, uncorrelated data $s_{D_p} = \sqrt{\frac{pq}{N_1} + \frac{pq}{N_2}}$	138
12.12	Alternate formula for the standard error of the difference between two proportions, small samples, uncorrelated data $s_{D_p} = \sqrt{pq\left(\frac{1}{N_1} + \frac{1}{N_2}\right)}$	138
12.13	Standard error of the difference between two proportions, correlated data $s_{D_p} = \sqrt{s_{p_1}^2 + s_{p_2}^2 - 2(r_{12})(s_{p_1})(s_{p_2})}$	139
12.14	The test of the difference between two proportions, correlated data, not requiring the computation of r $\tilde{z} = \sqrt{\frac{(a - d)^2}{a + d}}$	139
13.1	Standard error of r when the null hypothesis is made that $r = 0$ $s_{r_0} = \frac{1}{\sqrt{N - 1}}$	142
13.2	Small sample formula for the standard error of r when the null hypothesis is made that $r = 0$ $s_{r_0} = \sqrt{\frac{1 - r^2}{N - 2}}$	143
13.3	Standard error of Fisher's Z-statistic $s_Z = \frac{1}{\sqrt{N - 3}}$	144
13.4	Standard error of the difference between two Fisher Z-statistics $s_{D_Z} = \sqrt{s_{Z_1}^2 + s_{Z_2}^2}$	145

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13.5 Alternate formula for the standard error of the difference between two Fisher Z-statistics 145

$$s_{D_z} = \sqrt{\frac{1}{N_1 - 3} + \frac{1}{N_2 - 3}}$$

13.6 t-test for the significance of the difference between two r's correlated data 145

$$t = \frac{(r_{12} - r_{13})\sqrt{(N - 3)(1 + r_{23})}}{\sqrt{2(1 - r_{12}{}^2 - r_{13}{}^2 - r_{23}{}^2 + 2r_{12}r_{13}r_{23})}}$$

14.1 Chi square, general formula 147

$$\chi^2 = \sum\frac{(O - E)^2}{E}$$

14.2 Degrees of freedom in chi square 149

$$df = (r - 1)(c - 1)$$

14.3 Chi square formula for a 2 × 2 table using the obtained frequencies 149

$$\chi^2 = \frac{N[(ad) - (bc)]^2}{(k)(l)(m)(n)}$$

14.4 Yates' correction applied to formula 14.1 150

$$\chi^2 = \sum\frac{(|O - E| - .5)^2}{E}$$

14.5 Yates' correction in formula 14.3 150

$$\chi^2 = \frac{N\left(|ad - bc| - \frac{N}{2}\right)^2}{(k)(l)(m)(n)}$$

15.1 Equation for the total sum of squares 160

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{N}$$

15.2 Equation for the "between" sum of squares 161

$$\Sigma x^2 = \Sigma(\bar{X} - \bar{X}_T)n$$

15.3 Alternate equation for the "between" sum of squares 161

$$\Sigma x^2 = \left[\sum\frac{(\Sigma X)^2}{n}\right] - \frac{(\Sigma X_T)^2}{N}$$

15.4 Equation for the "within" sum of squares 161

$$\Sigma x^2 = \Sigma X^2 - \frac{(\Sigma X)^2}{n}$$

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16.2	*t*-test for the point-biserial *r* $t = \frac{r_{pb}\sqrt{N-2}}{\sqrt{1 - r_{pb}^2}}$	171
16.3	Biserial correlation coefficient $r_b = \frac{\bar{X}_p - \bar{X}_t}{s_t}\left(\frac{p}{y}\right)$	173
16.4	The fourfold of phi coefficient $\Phi = \frac{(ad - bc)}{\sqrt{(k)(l)(m)(n)}}$	175
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16.6	Relationships of chi square and phi $\chi^2 = N\Phi^2$	175
16.7	Coefficient of contingency $C = \sqrt{\frac{\chi^2}{N + \chi^2}}$	177
16.8	Spearman rank-order correlation coefficient $\rho = 1 - \frac{6\Sigma D^2}{N(N^2 - 1)}$	179
16.9	*t*-test for testing the significance of rho $t = \frac{\rho}{\sqrt{1 - \rho^2}}\sqrt{N - 2}$	179
16.10	Checking the sum of ranks in Kendall's *W* Total sum of ranks $= \frac{mN(N + 1)}{2}$	180
16.11	Coefficient of concordance, *W* $W = \frac{12\Sigma D^2}{m^2(N)(N^2 - 1)}$	181

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16.12 Eta squared 182

$$\eta_{yx}^2 = \frac{\Sigma y_b^2}{\Sigma y_t^2}$$

16.13 Correlation ratio, eta 182

$$\eta_{xy} = \sqrt{\frac{\Sigma y_b^2}{\Sigma y_t^2}}$$

16.14 Correlation ratio, operational formula 184

$$\eta_{yx}^2 = \frac{\sum\left[\frac{(\Sigma y')^2}{f_x}\right] - \frac{[\Sigma(\Sigma y')]^2}{N}}{\Sigma fy'^2 - \frac{(\Sigma fy')^2}{N}}$$

16.15 Partial r 186

$$r_{12.3} = \frac{r_{12} - (r_{13}r_{23})}{\sqrt{(1 - r_{13}^2)(1 - r_{23}^2)}}$$

16.16 Multiple correlation coefficient, R, three variables 188

$$R_{1.23} = \sqrt{\frac{r_{12}^2 + r_{13}^2 - (2r_{12}r_{13}r_{23})}{1 - r_{23}^2}}$$

16.17 Standard error of estimate of R 188

$$S_{1.23} = s_1\sqrt{1 - R^2}$$

17.1 Components of a single score 190

$$X = X_t + X_e$$

17.2 Breakdown of total test variance into component parts 191

$$s^2 = s_t^2 + s_e^2$$

17.3 Equation 17.2 with each term divided by s^2 191

$$\frac{s^2}{s^2} = \frac{s_t^2}{s^2} + \frac{s_e^2}{s^2}$$

17.4 Reliability equation 191

$$r_{tt} = \frac{s_t^2}{s^2}$$

17.5 Alternate equation for reliability 191

$$r_{tt} = 1 - \frac{s_e^2}{s^2}$$

17.6 Spearman-Brown formula to use with the split-half reliability technique 193

$$r_{tt} = \frac{2r_{oe}}{1 + r_{oe}}$$

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17.10	Standard error of estimate in predicting Y from X $s_{yx} = s_y\sqrt{1 - r_{xy}{}^2}$	198
17.11	Standard error of estimate in predicting X from Y $s_{xy} = s_x\sqrt{1 - r_{xy}{}^2}$	199
17.12	Relationship of r and k $r^2 + k^2 = 1$	199
17.13	k $k = \sqrt{1 - r^2}$	199
17.14	Index of forecasting efficiency $E = (1 - k)100$	200
17.15	Correction for attenuation in both test and criterion scores $r_c = \frac{r_{xy}}{\sqrt{r_{xx}r_{yy}}}$	200
17.16	Correction for attenuation in the criterion scores only $r_c = \frac{r_{xy}}{\sqrt{r_{yy}}}$	201
17.17	Formula for correction for guessing $\text{Score} = R - \frac{W}{N - 1}$	202
18.1	$\bar{T}$, sum of the ranks with the least frequent sign in Wilcoxon's matched-pairs signed-ranks test $\bar{T} = \frac{N(N + 1)}{4}$	210
18.2	$\bar{T}$ when N is greater than 25 $\bar{T} = \frac{(N)(N + 1)}{4}$	210

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18.3 s of ranks in Wilcoxon's test 210

$$s = \sqrt{\frac{(2N + 1)\bar{T}}{6}} \quad \text{or} \quad \sqrt{\frac{N(N + 1)(2N + 1)}{24}}$$

18.4 Computation of U_1 in the Mann-Whitney U-test 214

$$U_1 = N_1N_2 + \frac{N_1(N_1 + 1)}{2} - \Sigma R_x$$

18.5 $\tilde{z}$-ratio for the Mann-Whitney U-test 214

$$\tilde{z} = \frac{U_1 - \frac{N_1N_2}{2}}{\sqrt{\frac{N_1N_2(N_1 + N_2 + 1)}{12}}}$$

18.6 $\tilde{z}$-ratio for the Wald-Wolfowitz runs test 216

$$\tilde{z} = \frac{\left(r - \frac{2N_1N_2}{N_1 + N_2} + 1\right) - .5}{\frac{2N_1N_2(2N_1N_2 - N_1 - N_2)}{(N_1 + N_2)^2(N_1 + N_2 - 1)}}$$

18.7 Kruskal-Wallis H-test 217

$$H = \left[\frac{12}{N(N + 1)}\right]\left[\sum\frac{R_i{}^2}{N_i}\right] - 3(N + 1)$$

ANSWERS TO EXERCISES

Chapter 2

1. (a) 18.83
   (b) 109.67
   (c) 1.0974
   (d) -17.86
   (e) -1.0859
   (f) 48.13
   (g) -10.5022
   (h) 75.00

2. (a) 365.36
   (b) .074
   (c) .000056
   (d) $-.666$
   (e) -913.78
   (f) .03072
   (g) -7.77
   (h) 5.92

3. (a) 40
   (b) 9000
   (c) .025
   (d) -50.5
   (e) -360
   (f) 70
   (g) -10.058
   (h) 4.04

4. (a) 3.75
   (b) 3.875
   (c) 4.42
   (d) 2.67
   (e) 3.167
   (f) .686
   (g) -1.25
   (h) .56
   (i) 1.63
   (j) 24.25
   (k) 720
   (l) 40.607
   (m) 0
   (n) 64
   (o) -44
   (p) a^2b
   (q) .44
   (r) $x^2 + 2xy + y^2$
   (s) 3
   (t) fx^2
   (u) x^4y^3z

5. (a) 57
   (b) 175
   (c) 554
   (d) 27.89
   (e) 94.2
   (f) .0095
   (g) .0158
   (h) 3.0
   (i) .33
   (j) $.7\sqrt{-1}$
   (k) 276.5

6. (a) 14.4
   (b) 24.3
   (c) 17.8
   (d) .1
   (e) 1
   (f) 90.2
   (g) 88.0
   (h) 56.4
   (i) 48.6
   (j) .1

7. (a) 5
   (b) 4
   (c) 2
   (d) 2
   (e) 7
   (f) 1
   (g) 2
   (h) 1

Chapter 3

1. (a) $-.5, 4.5; 5$
   (b) $-2.5, -8.5; 6$
   (c) $1.5, 4.5; 3$
   (d) $-2.5, 4.5; 3$
   (e) $97.5, 103.5; 6$
   (f) $1.45, 3.55; 2.1$
   (g) $4.5, 10.5; 6$
   (h) $.495, .755; .26$

2. (a) An interval size of from 6 to 10 would work
   (b) .1
   (c) 2
   (d) 1

Chapter 4

1. $\bar{X} = 41.9$; Mdn = 41.7; Mo = 40

2. (a) $\bar{X} = 8.6$; Mdn = 8.5; Mo = 8 or 11

3. (a) $\bar{X} = 55.4$

4.

	$\bar{X}$	*Mdn*	*Mo*	C_{25}
Class I	42.1	43.4	48	32.8
Class II	41	42.5	48	31.8
Total	41.5	43	48	32.6

5. (a) 104 (using $i = 3$ and the bottom interval 87–89)
   (b) 102.2
   (d) 100
   (e) $C_{10} = 93.7$ $C_{33} = 100$ $C_{80} = 112.7$
   $C_{25} = 99$ $C_{75} = 111.5$ $C_{95} = 116.2$

6.

	$\bar{X}$	*Mdn*	*Mo*
(a)	29.6	30	—
(b)	25.8	27.5	30
(c)	11.3	10.8	10
(d)	15.2	14.2	14
(e)	24.4	24	—
(f)	6.3	6.2	6

7. 57

Chapter 5

1. (a) 13

2. (a) 60

3. (a) 11.9

4. (a) 2.1

5. (a) 15

6. (a) 4.0
   (b) .975

7. (a) 85.2
   (b) 12.8

8. $\bar{X} = 517.6$, $s = 63.3$

9. 1. $s = 7.4$
   2. $s = 3.1$
   3. $s = 10.1$
   4. Class I, $s = 14.1$
      Class II, $s = 15.3$
      Total, $s = 14.6$
   5. $s = 8.3$
   6. (a) $s = 3.3$
      (b) $s = 6.2$
      (c) $s = 2.6$
      (d) $s = 2.6$
      (e) $s = 3.5$
      (f) $s = 2.5$

Chapter 6

1. $\bar{X} = 26.1$, $s = 10.7$

2. -1.97; $-.76$; .38; 1.22; 1.86; 2.33

3. 30.3; 42.4; 53.8; 62.2; 68.6; 73.3

Chapter 7

2. .1064; .0596; .4987; .3443; .4932; .2881; .0199; .2422

3. No. cases: 118; 168; 39; 48; 298; 64; 156; 223
   Centile points: 61; 44; 99; 84; 1; 79; 48; 26

5. (b) 48.2%; 36.9%; 22.2%; 35.8%; 46.4%
   (c) 43.5%, 57 cases; 12.88%, 17 cases; 30.75%, 40 cases; 3.69%, 5 cases
   (d) 25; 22
   (e) 56.3; 48.3; 35.9; 29.3

Chapter 8

4. .92

5. .71

6. .81

7. .825

Chapter 9

1. (a) 7; (b) 24; (c) -7; (d) 50.88

3. 102.6

4. 3.7

5. (b) $b_{yx} = .70$, $a_{yx} = -7.0$
   $b_{xy} = 1.2$, $a_{xy} = -3.1$
   (d) $s_{yx} = 12.8$, $s_{xy} = 9.7$
   (e) 81

Chapter 10

2. (a) .077
   (b) .25
   (c) .000125
   (d) .5; .5
   (e) .204

3. (a) .031
   (b) .028
   (c) .5

5. $p = .19$

6. (a) 8
   (b) $p = .0463$
   (c) $p = .084$

7. Coca Cola $p = .08$; Pepsi Cola $p = .22$

8. (a) $p = .022$
   (b) $p = .05$
   (c) $p = .00000029$

Chapter 11

5. (a) 2
   (b) $p = .0228$ or 228 in 10,000
   (c) 5%, 61.1–68.9; 1%, 59.8–70.2

6. (a) 2
   (b) .179

Chapter 12

1. $z = 4.12$
2. $z = 2.26$
3. $t = 1.354$
4. $t = .760$
5. $z = 1.61$
6. $t = 4.17$
7. $t = 3.17$

9.

Group I	21.7–26.5	20.9–27.3
Group II	20.2–25.4	19.2–26.3

Chapter 13

1. $t = 1.51$
2. .1
3. $z = .83$
4. $t = -.857$
5. .79

Chapter 14

1. $\chi^2 = 5.71$
2. (a) $\chi^2 = 5.76$
   (b) $\chi^2 = 2.77$
3. Item 1 $\chi^2 = 1.66$
   Item 2 $\chi^2 = 13.20$
4. $\chi^2 = 2.314$
5. $\chi^2 = 3.476$
6. $\chi^2 = 16.66$
7. $\chi^2 = 5.901$
9. $\chi^2 = 11.16$
10. (b) 6
    (c) $\chi^2 = 5.882$
    (d) $\chi^2 = 27.66$
11. $\chi^2 = 13.09$ by combining the two "probable" groups into one

Chapter 15

1. (a) $F = .302$
2. $F = .54$
6. $F = .167$

Chapter 16

1. (a) .54
   (b) 8.96
2. .71
3. .40
4. .59
5. .31
6. .92
7. .79
8. .24
9. .45

Chapter 17

1. (a) .70
   (b) .77
   (c) 3.32
   (d) 1.58, using the Kuder-Richardson reliability formula
2. (a) .91
   (b) .82
   (c) 6
3. (a) 7.4
   (c) 12.3%
5. .72

Chapter 18

1. $\tilde{z} = 1.81$
2. Reject null hypothesis at 1% level
3. $\chi^2 = .808$
4. $\chi^2 = 3.17$
5. $U = 86.0$
6. $U = 80.5$
7. $\tilde{z} = -1.86$
8. Not significant
9. $H = 3.28$

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TABLES

TABLE I. Squares, Square Roots and Reciprocals of Integers from 1 to 1000

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
1	1	1.0000	1.000000	1.0000
2	4	1.4142	.500000	.7071
3	9	1.7321	.333333	.5774
4	16	2.0000	.250000	.5000
5	25	2.2361	.200000	.4472
6	36	2.4495	.166667	.4082
7	49	2.6458	.142857	.3780
8	64	2.8284	.125000	.3536
9	81	3.0000	.111111	.3333
10	100	3.1623	.100000	.3162
11	121	3.3166	.090909	.3015
12	144	3.4641	.083333	.2887
13	169	3.6056	.076923	.2774
14	196	3.7417	.071429	.2673
15	225	3.8730	.066667	.2582
16	256	4.0000	.062500	.2500
17	289	4.1231	.058824	.2425
18	324	4.2426	.055556	.2357
19	361	4.3589	.052632	.2294
20	400	4.4721	.050000	.2236
21	441	4.5826	.047619	.2182
22	484	4.6904	.045455	.2132
23	529	4.7958	.043478	.2085
24	576	4.8990	.041667	.2041
25	625	5.0000	.040000	.2000
26	676	5.0990	.038462	.1961
27	729	5.1962	.037037	.1925
28	784	5.2915	.035714	.1890
29	841	5.3852	.034483	.1857
30	900	5.4772	.033333	.1826
31	961	5.5678	.032258	.1796
32	1024	5.6569	.031250	.1768
33	1089	5.7446	.030303	.1741
34	1156	5.8310	.029412	.1715
35	1225	5.9161	.028571	.1690

SOURCE: J. G. Peatman: *Descriptive and Sampling Statistics.* New York: Harper, 1947. Reprinted by permission of the publisher.

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
36	1296	6.0000	.027778	.1667
37	1369	6.0828	.027027	.1644
38	1444	6.1644	.026316	.1622
39	1521	6.2450	.025641	.1601
40	1600	6.3246	.025000	.1581
41	1681	6.4031	.024390	.1562
42	1764	6.4807	.023810	.1543
43	1849	6.5574	.023256	.1525
44	1936	6.6332	.022727	.1508
45	2025	6.7082	.022222	.1491
46	2116	6.7823	.021739	.1474
47	2209	6.8557	.021277	.1459
48	2304	6.9282	.020833	.1443
49	2401	7.0000	.020408	.1429
50	2500	7.0711	.020000	.1414
51	2601	7.1414	.019608	.1400
52	2704	7.2111	.019231	.1387
53	2809	7.2801	.018868	.1374
54	2916	7.3485	.018519	.1361
55	3025	7.4162	.018182	.1348
56	3136	7.4833	.017857	.1336
57	3249	7.5498	.017544	.1325
58	3364	7.6158	.017241	.1313
59	3481	7.6811	.016949	.1302
60	3600	7.7460	.016667	.1291
61	3721	7.8102	.016393	.1280
62	3844	7.8740	.016129	.1270
63	3969	7.9373	.015873	.1260
64	4096	8.0000	.015625	.1250
65	4225	8.0623	.015385	.1240
66	4356	8.1240	.015152	.1231
67	4489	8.1854	.014925	.1222
68	4624	8.2462	.014706	.1213
69	4761	8.3066	.014493	.1204
70	4900	8.3666	.014286	.1195
71	5041	8.4261	.014085	.1187
72	5184	8.4853	.013889	.1179
73	5329	8.5440	.013699	.1170
74	5476	8.6023	.013514	.1162
75	5625	8.6603	.013333	.1155
76	5776	8.7178	.013158	.1147
77	5929	8.7750	.012987	.1140
78	6084	8.8318	.012821	.1132
79	6241	8.8882	.012658	.1125
80	6400	8.9443	.012500	.1118
81	6561	9.0000	.012346	.1111
82	6724	9.0554	.012195	.1104
83	6889	9.1104	.012048	.1098
84	7056	9.1652	.011905	.1091
85	7225	9.2195	.011765	.1085

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
86	7396	9.2736	.011628	.1078
87	7569	9.3274	.011494	.1072
88	7744	9.3808	.011364	.1066
89	7921	9.4340	.011236	.1060
90	8100	9.4868	.011111	.1054
91	8281	9.5394	.010989	.1048
92	8464	9.5917	.010870	.1043
93	8649	9.6437	.010753	.1037
94	8836	9.6954	.010638	.1031
95	9025	9.7468	.010526	.1026
96	9216	9.7980	.010417	.1021
97	9409	9.8489	.010309	.1015
98	9604	9.8995	.010204	.1010
99	9801	9.9499	.010101	.1005
100	10000	10.0000	.010000	.1000
101	10201	10.0499	.009901	.0995
102	10404	10.0995	.009804	.0990
103	10609	10.1489	.009709	.0985
104	10816	10.1980	.009615	.0981
105	11025	10.2470	.009524	.0976
106	11236	10.2956	.009434	.0971
107	11449	10.3441	.009346	.0967
108	11664	10.3923	.009259	.0962
109	11881	10.4403	.009174	.0958
110	12100	10.4881	.009091	.0953
111	12321	10.5357	.009009	.0949
112	12544	10.5830	.008929	.0945
113	12769	10.6301	.008850	.0941
114	12996	10.6771	.008772	.0937
115	13225	10.7238	.008696	.0933
116	13456	10.7703	.008621	.0928
117	13689	10.8167	.008547	.0925
118	13924	10.8628	.008475	.0921
119	14161	10.9087	.008403	.0917
120	14400	10.9545	.008333	.0913
121	14641	11.0000	.008264	.0909
122	14884	11.0454	.008197	.0905
123	15129	11.0905	.008130	.0902
124	15376	11.1355	.008065	.0898
125	15625	11.1803	.008000	.0894
126	15876	11.2250	.007937	.0891
127	16129	11.2694	.007874	.0887
128	16384	11.3137	.007813	.0884
129	16641	11.3578	.007752	.0880
130	16900	11.4018	.007692	.0877
131	17161	11.4455	.007634	.0874
132	17424	11.4891	.007576	.0870
133	17689	11.5326	.007519	.0867
134	17956	11.5758	.007463	.0864
135	18225	11.6190	.007407	.0861

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
136	18496	11.6619	.007353	.0857
137	18769	11.7047	.007299	.0854
138	19044	11.7473	.007246	.0851
139	19321	11.7898	.007194	.0848
140	19600	11.8322	.007143	.0845
141	19881	11.8743	.007092	.0842
142	20164	11.9164	.007042	.0839
143	20449	11.9583	.006993	.0836
144	20736	12.0000	.006944	.0833
145	21025	12.0416	.006897	.0830
146	21316	12.0830	.006849	.0828
147	21609	12.1244	.006803	.0825
148	21904	12.1655	.006757	.0822
149	22201	12.2066	.006711	.0819
150	22500	12.2474	.006667	.0816
151	22801	12.2882	.006623	.0814
152	23104	12.3288	.006579	.0811
153	23409	12.3693	.006536	.0808
154	23716	12.4097	.006494	.0806
155	24025	12.4499	.006452	.0803
156	24336	12.4900	.006410	.0801
157	24649	12.5300	.006369	.0798
158	24964	12.5698	.006329	.0796
159	25281	12.6095	.006289	.0793
160	25600	12.6491	.006250	.0791
161	25921	12.6886	.006211	.0788
162	26244	12.7279	.006173	.0786
163	26569	12.7671	.006135	.0783
164	26896	12.8062	.006098	.0781
165	27225	12.8452	.006061	.0778
166	27556	12.8841	.006024	.0776
167	27889	12.9228	.005988	.0774
168	28224	12.9615	.005952	.0772
169	28561	13.0000	.005917	.0769
170	28900	13.0384	.005882	.0767
171	29241	13.0767	.005848	.0765
172	29584	13.1149	.005814	.0762
173	29929	13.1529	.005780	.0760
174	30276	13.1909	.005747	.0758
175	30625	13.2288	.005714	.0756
176	30976	13.2665	.005682	.0754
177	31329	13.3041	.005650	.0752
178	31684	13.3417	.005618	.0750
179	32041	13.3791	.005587	.0747
180	32400	13.4164	.005556	.0745
181	32761	13.4536	.005525	.0743
182	33124	13.4907	.005495	.0741
183	33489	13.5277	.005464	.0739
184	33856	13.5647	.005435	.0737
185	34225	13.6015	.005405	.0735

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
186	34596	13.6382	.005376	.0733
187	34969	13.6748	.005348	.0731
188	35344	13.7113	.005319	.0729
189	35721	13.7477	.005291	.0727
190	36100	13.7840	.005263	.0725
191	36481	13.8203	.005236	.0724
192	36864	13.8564	.005208	.0722
193	37249	13.8924	.005181	.0720
194	37636	13.9284	.005155	.0718
195	38025	13.9642	.005128	.0716
196	38416	14.0000	.005102	.0714
197	38809	14.0357	.005076	.0712
198	39204	14.0712	.005051	.0711
199	39601	14.1067	.005025	.0709
200	40000	14.1421	.005000	.0707
201	40401	14.1774	.004975	.0705
202	40804	14.2127	.004950	.0704
203	41209	14.2478	.004926	.0702
204	41616	14.2829	.004902	.0700
205	42025	14.3178	.004878	.0698
206	42436	14.3527	.004854	.0697
207	42849	14.3875	.004831	.0695
208	43264	14.4222	.004808	.0693
209	43681	14.4568	.004785	.0692
210	44100	14.4914	.004762	.0690
211	44521	14.5258	.004739	.0688
212	44944	14.5602	.004717	.0687
213	45369	14.5945	.004695	.0685
214	45796	14.6287	.004673	.0684
215	46225	14.6629	.004651	.0682
216	46656	14.6969	.004630	.0680
217	47089	14.7309	.004608	.0679
218	47524	14.7648	.004587	.0677
219	47961	14.7986	.004566	.0676
220	48400	14.8324	.004545	.0674
221	48841	14.8661	.004525	.0673
222	49284	14.8997	.004505	.0671
223	49729	14.9332	.004484	.0670
224	50176	14.9666	.004464	.0668
225	50625	15.0000	.004444	.0667
226	51076	15.0333	.004425	.0665
227	51529	15.0665	.004405	.0664
228	51984	15.0997	.004386	.0662
229	52441	15.1327	.004367	.0661
230	52900	15.1658	.004348	.0659
231	53361	15.1987	.004329	.0658
232	53824	15.2315	.004310	.0657
233	54289	15.2643	.004292	.0655
234	54756	15.2971	.004274	.0654
235	55225	15.3297	.004255	.0652

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
236	55696	15.3623	.004237	.0651
237	56169	15.3948	.004219	.0650
238	56644	15.4272	.004202	.0648
239	57121	15.4596	.004184	.0647
240	57600	15.4919	.004167	.0645
241	58081	15.5242	.004149	.0644
242	58564	15.5563	.004132	.0643
243	59049	15.5885	.004115	.0642
244	59536	15.6205	.004098	.0640
245	60025	15.6525	.004082	.0639
246	60516	15.6844	.004065	.0638
247	61009	15.7162	.004049	.0636
248	61504	15.7480	.004032	.0635
249	62001	15.7797	.004016	.0634
250	62500	15.8114	.004000	.0632
251	63001	15.8430	.003984	.0631
252	63504	15.8745	.003968	.0630
253	64009	15.9060	.003953	.0629
254	64516	15.9374	.003937	.0627
255	65025	15.9687	.003922	.0626
256	65536	16.0000	.003906	.0625
257	66049	16.0312	.003891	.0624
258	66564	16.0624	.003876	.0623
259	67081	16.0935	.003861	.0621
260	67600	16.1245	.003846	.0620
261	68121	16.1555	.003831	.0619
262	68644	16.1864	.003817	.0618
263	69169	16.2173	.003802	.0617
264	69696	16.2481	.003788	.0615
265	70225	16.2788	.003774	.0614
266	70756	16.3095	.003759	.0613
267	71289	16.3401	.003745	.0612
268	71824	16.3707	.003731	.0611
269	72361	16.4012	.003717	.0610
270	72900	16.4317	.003704	.0609
271	73441	16.4621	.003690	.0607
272	73984	16.4924	.003676	.0606
273	74529	16.5227	.003663	.0605
274	75076	16.5529	.003650	.0604
275	75625	16.5831	.003636	.0603
276	76176	16.6132	.003623	.0602
277	76729	16.6433	.003610	.0601
278	77284	16.6733	.003597	.0600
279	77841	16.7033	.003584	.0599
280	78400	16.7332	.003571	.0598
281	78961	16.7631	.003559	.0597
282	79524	16.7929	.003546	.0595
283	80089	16.8226	.003534	.0594
284	80656	16.8523	.003521	.0593
285	81225	16.8819	.003509	.0592

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
286	81796	16.9115	.003497	.0591
287	82369	16.9411	.003484	.0590
288	82944	16.9706	.003472	.0589
289	83521	17.0000	.003460	.0588
290	84100	17.0294	.003448	.0587
291	84681	17.0587	.003436	.0586
292	85264	17.0880	.003425	.0585
293	85849	17.1172	.003413	.0584
294	86436	17.1464	.003401	.0583
295	87025	17.1756	.003390	.0582
296	87616	17.2047	.003378	.0581
297	88209	17.2337	.003367	.0580
298	88804	17.2627	.003356	.0579
299	89401	17.2916	.003344	.0578
300	90000	17.3205	.003333	.0577
301	90601	17.3494	.003322	.0576
302	91204	17.3781	.003311	.0575
303	91809	17.4069	.003300	.0574
304	92416	17.4356	.003289	.0574
305	93025	17.4642	.003279	.0573
306	93636	17.4929	.003268	.0572
307	94249	17.5214	.003257	.0571
308	94864	17.5499	.003247	.0570
309	95481	17.5784	.003236	.0569
310	96100	17.6068	.003226	.0568
311	96721	17.6352	.003215	.0567
312	97344	17.6635	.003205	.0566
313	97969	17.6918	.003195	.0565
314	98596	17.7200	.003185	.0564
315	99225	17.7482	.003175	.0563
316	99856	17.7764	.003165	.0563
317	100489	17.8045	.003155	.0562
318	101124	17.8326	.003145	.0561
319	101761	17.8606	.003135	.0560
320	102400	17.8885	.003125	.0559
321	103041	17.9165	.003115	.0558
322	103684	17.9444	.003106	.0557
323	104329	17.9722	.003096	.0556
324	104976	18.0000	.003086	.0556
325	105625	18.0278	.003077	.0555
326	106276	18.0555	.003067	.0554
327	106929	18.0831	.003058	.0553
328	107584	18.1108	.003049	.0552
329	108241	18.1384	.003040	.0551
330	108900	18.1659	.003030	.0550
331	109561	18.1934	.003021	.0550
332	110224	18.2209	.003012	.0549
333	110889	18.2483	.003003	.0548
334	111556	18.2757	.002994	.0547
335	112225	18.3030	.002985	.0546

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
336	112896	18.3303	.002976	.0546
337	113569	18.3576	.002967	.0545
338	114244	18.3848	.002959	.0544
339	114921	18.4120	.002950	.0543
340	115600	18.4391	.002941	.0542
341	116281	18.4662	.002933	.0542
342	116964	18.4932	.002924	.0541
343	117649	18.5203	.002915	.0540
344	118336	18.5472	.002907	.0539
345	119025	18.5742	.002899	.0538
346	119716	18.6011	.002890	.0538
347	120409	18.6279	.002882	.0537
348	121104	18.6548	.002874	.0536
349	121801	18.6815	.002865	.0535
350	122500	18.7083	.002857	.0535
351	123201	18.7350	.002849	.0534
352	123904	18.7617	.002841	.0533
353	124609	18.7883	.002833	.0532
354	125316	18.8149	.002825	.0531
355	126025	18.8414	.002817	.0531
356	126736	18.8680	.002809	.0530
357	127449	18.8944	.002801	.0529
358	128164	18.9209	.002793	.0529
359	128881	18.9473	.002786	.0528
360	129600	18.9737	.002778	.0527
361	130321	19.0000	.002770	.0526
362	131044	19.0263	.002762	.0526
363	131769	19.0526	.002755	.0525
364	132496	19.0788	.002747	.0524
365	133225	19.1050	.002740	.0523
366	133956	19.1311	.002732	.0523
367	134689	19.1572	.002725	.0522
368	135424	19.1833	.002717	.0521
369	136161	19.2094	.002710	.0521
370	136900	19.2354	.002703	.0520
371	137641	19.2614	.002695	.0519
372	138384	19.2873	.002688	.0518
373	139129	19.3132	.002681	.0518
374	139876	19.3391	.002674	.0517
375	140625	19.3649	.002667	.0516
376	141376	19.3907	.002660	.0516
377	142129	19.4165	.002653	.0515
378	142884	19.4422	.002646	.0514
379	143641	19.4679	.002639	.0514
380	144400	19.4936	.002632	.0513
381	145161	19.5192	.002625	.0512
382	145924	19.5448	.002618	.0512
383	146689	19.5704	.002611	.0511
384	147456	19.5959	.002604	.0510
385	148225	19.6214	.002597	.0510

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
386	148996	19.6469	.002591	.0509
387	149769	19.6723	.002584	.0508
388	150544	19.6977	.002577	.0508
389	151321	19.7231	.002571	.0507
390	152100	19.7484	.002564	.0506
391	152881	19.7737	.002558	.0506
392	153664	19.7990	.002551	.0505
393	154449	19.8242	.002545	.0504
394	155236	19.8494	.002538	.0504
395	156025	19.8746	.002532	.0503
396	156816	19.8997	.002525	.0503
397	157609	19.9249	.002519	.0502
398	158404	19.9499	.002513	.0501
399	159201	19.9750	.002506	.0501
400	160000	20.0000	.002500	.0500
401	160801	20.0250	.002494	.0499
402	161604	20.0499	.002488	.0499
403	162409	20.0749	.002481	.0498
404	163216	20.0998	.002475	.0498
405	164025	20.1246	.002469	.0497
406	164836	20.1494	.002463	.0496
407	165649	20.1742	.002457	.0496
408	166464	20.1990	.002451	.0495
409	167281	20.2237	.002445	.0494
410	168100	20.2485	.002439	.0494
411	168921	20.2731	.002433	.0493
412	169744	20.2978	.002427	.0493
413	170569	20.3224	.002421	.0492
414	171396	20.3470	.002415	.0491
415	172225	20.3715	.002410	.0491
416	173056	20.3961	.002404	.0490
417	173889	20.4206	.002398	.0490
418	174724	20.4450	.002392	.0489
419	175561	20.4695	.002387	.0489
420	176400	20.4939	.002381	.0488
421	177241	20.5183	.002375	.0487
422	178084	20.5426	.002370	.0487
423	178929	20.5670	.002364	.0486
424	179776	20.5913	.002358	.0486
425	180625	20.6155	.002353	.0485
426	181476	20.6398	.002347	.0485
427	182329	20.6640	.002342	.0484
428	183184	20.6882	.002336	.0483
429	184041	20.7123	.002331	.0483
430	184900	20.7364	.002326	.0482
431	185761	20.7605	.002320	.0482
432	186624	20.7846	.002315	.0481
433	187489	20.8087	.002309	.0481
434	188356	20.8327	.002304	.0480
435	189225	20.8567	.002299	.0479

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
436	190096	20.8806	.002294	.0479
437	190969	20.9045	.002288	.0478
438	191844	20.9284	.002283	.0478
439	192721	20.9523	.002278	.0477
440	193600	20.9762	.002273	.0477
441	194481	21.0000	.002268	.0476
442	195364	21.0238	.002262	.0476
443	196249	21.0476	.002257	.0475
444	197136	21.0713	.002252	.0475
445	198025	21.0950	.002247	.0474
446	198916	21.1187	.002242	.0474
447	199809	21.1424	.002237	.0473
448	200704	21.1660	.002232	.0472
449	201601	21.1896	.002227	.0472
450	202500	21.2132	.002222	.0471
451	203401	21.2368	.002217	.0471
452	204304	21.2603	.002212	.0470
453	205209	21.2838	.002208	.0470
454	206116	21.3073	.002203	.0469
455	207025	21.3307	.002198	.0469
456	207936	21.3542	.002193	.0468
457	208849	21.3776	.022188	.0468
458	209764	21.4009	.002183	.0467
459	210681	21.4243	.002179	.0467
460	211600	21.4476	.002174	.0466
461	212521	21.4709	.002169	.0466
462	213444	21.4942	.002165	.0465
463	214369	21.5174	.002160	.0465
464	215296	21.5407	.002155	.0464
465	216225	21.5639	.002151	.0464
466	217156	21.5870	.002146	.0463
467	218089	21.6102	.002141	.0463
468	219024	21.6333	.002137	.0462
469	219961	21.6564	.002132	.0462
470	220900	21.6795	.002128	.0461
471	221841	21.7025	.002123	.0461
472	222784	21.7256	.002119	.0460
473	223729	21.7486	.002114	.0460
474	224676	21.7715	.002110	.0459
475	225625	21.7945	.002105	.0459
476	226576	21.8174	.002101	.0458
477	227529	21.8403	.002096	.0458
478	228484	21.8632	.002092	.0457
479	229441	21.8861	.002088	.0457
480	230400	21.9089	.002083	.0456
481	231361	21.9317	.002079	.0456
482	232324	21.9545	.002075	.0455
483	233289	21.9773	.002070	.0455
484	234256	22.0000	.002066	.0455
485	235225	22.0227	.002062	.0454

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
386	148996	19.6469	.002591	.0509
387	149769	19.6723	.002584	.0508
388	150544	19.6977	.002577	.0508
389	151321	19.7231	.002571	.0507
390	152100	19.7484	.002564	.0506
391	152881	19.7737	.002558	.0506
392	153664	19.7990	.002551	.0505
393	154449	19.8242	.002545	.0504
394	155236	19.8494	.002538	.0504
395	156025	19.8746	.002532	.0503
396	156816	19.8997	.002525	.0503
397	157609	19.9249	.002519	.0502
398	158404	19.9499	.002513	.0501
399	159201	19.9750	.002506	.0501
400	160000	20.0000	.002500	.0500
401	160801	20.0250	.002494	.0499
402	161604	20.0499	.002488	.0499
403	162409	20.0749	.002481	.0498
404	163216	20.0998	.002475	.0498
405	164025	20.1246	.002469	.0497
406	164836	20.1494	.002463	.0496
407	165649	20.1742	.002457	.0496
408	166464	20.1990	.002451	.0495
409	167281	20.2237	.002445	.0494
410	168100	20.2485	.002439	.0494
411	168921	20.2731	.002433	.0493
412	169744	20.2978	.002427	.0493
413	170569	20.3224	.002421	.0492
414	171396	20.3470	.002415	.0491
415	172225	20.3715	.002410	.0491
416	173056	20.3961	.002404	.0490
417	173889	20.4206	.002398	.0490
418	174724	20.4450	.002392	.0489
419	175561	20.4695	.002387	.0489
420	176400	20.4939	.002381	.0488
421	177241	20.5183	.002375	.0487
422	178084	20.5426	.002370	.0487
423	178929	20.5670	.002364	.0486
424	179776	20.5913	.002358	.0486
425	180625	20.6155	.002353	.0485
426	181476	20.6398	.002347	.0485
427	182329	20.6640	.002342	.0484
428	183184	20.6882	.002336	.0483
429	184041	20.7123	.002331	.0483
430	184900	20.7364	.002326	.0482
431	185761	20.7605	.002320	.0482
432	186624	20.7846	.002315	.0481
433	187489	20.8087	.002309	.0481
434	188356	20.8327	.002304	.0480
435	189225	20.8567	.002299	.0479

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
436	190096	20.8806	.002294	.0479
437	190969	20.9045	.002288	.0478
438	191844	20.9284	.002283	.0478
439	192721	20.9523	.002278	.0477
440	193600	20.9762	.002273	.0477
441	194481	21.0000	.002268	.0476
442	195364	21.0238	.002262	.0476
443	196249	21.0476	.002257	.0475
444	197136	21.0713	.002252	.0475
445	198025	21.0950	.002247	.0474
446	198916	21.1187	.002242	.0474
447	199809	21.1424	.002237	.0473
448	200704	21.1660	.002232	.0472
449	201601	21.1896	.002227	.0472
450	202500	21.2132	.002222	.0471
451	203401	21.2368	.002217	.0471
452	204304	21.2603	.002212	.0470
453	205209	21.2838	.002208	.0470
454	206116	21.3073	.002203	.0469
455	207025	21.3307	.002198	.0469
456	207936	21.3542	.002193	.0468
457	208849	21.3776	.022188	.0468
458	209764	21.4009	.002183	.0467
459	210681	21.4243	.002179	.0467
460	211600	21.4476	.002174	.0466
461	212521	21.4709	.002169	.0466
462	213444	21.4942	.002165	.0465
463	214369	21.5174	.002160	.0465
464	215296	21.5407	.002155	.0464
465	216225	21.5639	.002151	.0464
466	217156	21.5870	.002146	.0463
467	218089	21.6102	.002141	.0463
468	219024	21.6333	.002137	.0462
469	219961	21.6564	.002132	.0462
470	220900	21.6795	.002128	.0461
471	221841	21.7025	.002123	.0461
472	222784	21.7256	.002119	.0460
473	223729	21.7486	.002114	.0460
474	224676	21.7715	.002110	.0459
475	225625	21.7945	.002105	.0459
476	226576	21.8174	.002101	.0458
477	227529	21.8403	.002096	.0458
478	228484	21.8632	.002092	.0457
479	229441	21.8861	.002088	.0457
480	230400	21.9089	.002083	.0456
481	231361	21.9317	.002079	.0456
482	232324	21.9545	.002075	.0455
483	233289	21.9773	.002070	.0455
484	234256	22.0000	.002066	.0455
485	235225	22.0227	.002062	.0454

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
486	236196	22.0454	.002058	.0454
487	237169	22.0681	.002053	.0453
488	238144	22.0907	.002049	.0453
489	239121	22.1133	.002045	.0452
490	240100	22.1359	.002041	.0452
491	241081	22.1585	.002037	.0451
492	242064	22.1811	.002033	.0451
493	243049	22.2036	.002028	.0450
494	244036	22.2261	.002024	.0450
495	245025	22.2486	.002020	.0449
496	246016	22.2711	.002016	.0448
497	247009	22.2935	.002012	.0449
498	248004	22.3159	.002008	.0449
499	249001	22.3383	.002004	.0448
500	250000	22.3607	.002000	.0447
501	251001	22.3830	.001996	.0447
502	252004	22.4054	.001992	.0446
503	253009	22.4277	.001988	.0446
504	254016	22.4499	.001984	.0445
505	255025	22.4722	.001980	.0445
506	256036	22.4944	.001976	.0445
507	257049	22.5167	.001972	.0444
508	258064	22.5389	.001969	.0444
509	259081	22.5610	.001965	.0443
510	260100	22.5832	.001961	.0443
511	261121	22.6053	.001957	.0442
512	262144	22.6274	.001953	.0442
513	263169	22.6495	.001949	.0442
514	264196	22.6716	.001946	.0441
515	265225	22.6936	.001942	.0441
516	266256	22.7156	.001938	.0440
517	267289	22.7376	.001934	.0440
518	268324	22.7596	.001931	.0439
519	269361	22.7816	.001927	.0439
520	270400	22.8035	.001923	.0439
521	271441	22.8254	.001919	.0438
522	272484	22.8473	.001916	.0438
523	273529	22.8692	.001912	.0437
524	274576	22.8910	.001908	.0437
525	275625	22.9129	.001905	.0436
526	276676	22.9347	.001901	.0436
527	277729	22.9565	.001898	.0436
528	278784	22.9783	.001894	.0435
529	279841	23.0000	.001890	.0435
530	280900	23.0217	.001887	.0434
531	281961	23.0434	.001883	.0434
532	283024	23.0651	.001880	.0434
533	284089	23.0868	.001876	.0433
534	285156	23.1084	.001873	.0433
535	286225	23.1301	.001869	.0432

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
536	287296	23.1517	.001866	.0432
537	288369	23.1733	.001862	.0432
538	289444	23.1948	.001859	.0431
539	290521	23.2164	.001855	.0431
540	291600	23.2379	.001852	.0430
541	292681	23.2594	.001848	.0430
542	293764	23.2809	.001845	.0430
543	294849	23.3024	.001842	.0429
544	295936	23.3238	.001838	.0429
545	297025	23.3452	.001835	.0428
546	298116	23.3666	.001832	.0428
547	299209	23.3880	.001828	.0428
548	300304	23.4094	.001825	.0427
549	301401	23.4307	.001821	.0427
550	302500	23.4521	.001818	.0426
551	303601	23.4734	.001815	.0426
552	304704	23.4947	.001812	.0426
553	305809	23.5160	.001808	.0425
554	306916	23.5372	.001805	.0425
555	308025	23.5584	.001802	.0424
556	309136	23.5797	.001799	.0424
557	310249	23.6008	.001795	.0424
558	311364	23.6220	.001792	.0423
559	312481	23.6432	.001789	.0423
560	313600	23.6643	.001786	.0423
561	314721	23.6854	.001783	.0422
562	315844	23.7065	.001779	.0422
563	316969	23.7276	.001776	.0421
564	318096	23.7487	.001773	.0421
565	319225	23.7697	.001770	.0421
566	320356	23.7908	.001767	.0420
567	321489	23.8118	.001764	.0420
568	322624	23.8328	.001761	.0420
569	323761	23.8537	.001757	.0419
570	324900	23.8747	.001754	.0419
571	326041	23.8956	.001751	.0418
572	327184	23.9165	.001748	.0418
573	328329	23.9374	.001745	.0418
574	329476	23.9583	.001742	.0417
575	330625	23.9792	.001739	.0417
576	331776	24.0000	.001736	.0417
577	332929	24.0208	.001733	.0416
578	334084	24.0416	.001730	.0416
579	335241	24.0624	.001727	.0416
580	336400	24.0832	.001724	.0415
581	337561	24.1039	.001721	.0415
582	338724	24.1247	.001718	.0415
583	339889	24.1454	.001715	.0414
584	341056	24.1661	.001712	.0414
585	342225	24.1868	.001709	.0413

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
586	343396	24.2074	.001706	.0413
587	344569	24.2281	.001704	.0413
588	345744	24.2487	.001701	.0412
589	346921	24.2693	.001698	.0412
590	348100	24.2899	.001695	.0412
591	349281	24.3105	.001692	.0411
592	350464	24.3311	.001689	.0411
593	351649	24.3516	.001686	.0411
594	352836	24.3721	.001684	.0410
595	354025	24.3926	.001681	.0410
596	355216	24.4131	.001678	.0410
597	356409	24.4336	.001675	.0409
598	357604	24.4540	.001672	.0409
599	358801	24.4745	.001669	.0409
600	360000	24.4949	.001667	.0408
601	361201	24.5153	.001664	.0408
602	362404	24.5357	.001661	.0408
603	363609	24.5561	.001658	.0407
604	364816	24.5764	.001656	.0407
605	366025	24.5967	.001653	.0407
606	367236	24.6171	.001650	.0406
607	368449	24.6374	.001647	.0406
608	369664	24.6577	.001645	.0406
609	370881	24.6779	.001642	.0405
610	372100	24.6982	.001639	.0405
611	373321	24.7184	.001637	.0405
612	374544	24.7386	.001634	.0404
613	375769	24.7588	.001631	.0404
614	376996	24.7790	.001629	.0404
615	378225	24.7992	.001626	.0403
616	379456	24.8193	.001623	.0403
617	380689	24.8395	.001621	.0403
618	381924	24.8596	.001618	.0402
619	383161	24.8797	.001616	.0402
620	384400	24.8998	.001613	.0402
621	385641	24.9199	.001610	.0401
622	386884	24.9399	.001608	.0401
623	388129	24.9600	.001605	.0401
624	389376	24.9800	.001603	.0400
625	390625	25.0000	.001600	.0400
626	391876	25.0200	.001597	.0400
627	393129	25.0400	.001595	.0399
628	394384	25.0599	.001592	.0399
629	395641	25.0799	.001590	.0399
630	396900	25.0998	.001587	.0398
631	398161	25.1197	.001585	.0398
632	399424	25.1396	.001582	.0398
633	400689	25.1595	.001580	.0397
634	401956	25.1794	.001577	.0397
635	403225	25.1992	.001575	.0397

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
636	404496	25.2190	.001572	.0397
637	405769	25.2389	.001570	.0396
638	407044	25.2587	.001567	.0396
639	408321	25.2784	.001565	.0396
640	409600	25.2982	.001563	.0395
641	410881	25.3180	.001560	.0395
642	412164	25.3377	.001558	.0395
643	413449	25.3574	.001555	.0394
644	414736	25.3772	.001553	.0394
645	416025	25.3969	.001550	.0394
646	417316	25.4165	.001548	.0393
647	418609	25.4362	.001546	.0393
648	419904	25.4558	.001543	.0393
649	421201	25.4755	.001541	.0393
650	422500	25.4951	.001538	.0392
651	423801	25.5147	.001536	.0392
652	425104	25.5343	.001534	.0392
653	426409	25.5539	.001531	.0391
654	427716	25.5734	.001529	.0391
655	429025	25.5930	.001527	.0391
656	430336	25.6125	.001524	.0390
657	431649	25.6320	.001522	.0390
658	432964	25.6515	.001520	.0390
659	434281	25.6710	.001517	.0390
660	435600	25.6905	.001515	.0389
661	436921	25.7099	.001513	.0389
662	438244	25.7294	.001511	.0389
663	439569	25.7488	.001508	.0388
664	440896	25.7682	.001506	.0388
665	442225	25.7876	.001504	.0388
666	443556	25.8070	.001502	.0387
667	444889	25.8263	.001499	.0387
668	446224	25.8457	.001497	.0387
669	447561	25.8650	.001495	.0387
670	448900	25.8844	.001493	.0386
671	450241	25.9037	.001490	.0386
672	451584	25.9230	.001488	.0386
673	452929	25.9422	.001486	.0385
674	454276	25.9615	.001484	.0385
675	455625	25.9808	.001481	.0385
676	456976	26.0000	.001479	.0385
677	458329	26.0192	.001477	.0384
678	459684	26.0384	.001475	.0384
679	461041	26.0576	.001473	.0384
680	462400	26.0768	.001471	.0383
681	463761	26.0960	.001468	.0383
682	465124	26.1151	.001466	.0383
683	466489	26.1343	.001464	.0383
684	467856	26.1534	.001462	.0382
685	469225	26.1725	.001460	.0382

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
686	470596	26.1916	.001458	.0382
687	471969	26.2107	.001456	.0382
688	473344	26.2298	.001453	.0381
689	474721	26.2488	.001451	.0381
690	476100	26.2679	.001449	.0381
691	477481	26.2869	.001447	.0380
692	478864	26.3059	.001445	.0380
693	480249	26.3249	.001443	.0380
694	481636	26.3439	.001441	.0380
695	483025	26.3629	.001439	.0379
696	484416	26.3818	.001437	.0379
697	485809	26.4008	.001435	.0379
698	487204	26.4197	.001433	.0379
699	488601	26.4386	.001431	.0378
700	490000	26.4575	.001429	.0378
701	491401	26.4764	.001427	.0378
702	492804	26.4953	.001425	.0377
703	494209	26.5141	.001422	.0377
704	495616	26.5330	.001420	.0377
705	497025	26.5518	.001418	.0377
706	498436	26.5707	.001416	.0376
707	499849	26.5895	.001414	.0376
708	501264	26.6083	.001412	.0376
709	502681	26.6271	.001410	.0376
710	504100	26.6458	.001408	.0375
711	505521	26.6646	.001406	.0375
712	506944	26.6833	.001404	.0375
713	508369	26.7021	.001403	.0375
714	509796	26.7208	.001401	.0374
715	511225	26.7395	.001399	.0374
716	512656	26.7582	.001397	.0374
717	514089	26.7769	.001395	.0373
718	515524	26.7955	.001393	.0373
719	516961	26.8142	.001391	.0373
720	518400	26.8328	.001389	.0373
721	519841	26.8514	.001387	.0372
722	521284	26.8701	.001385	.0372
723	522729	26.8887	.001383	.0372
724	524176	26.9072	.001381	.0372
725	525625	26.9258	.001379	.0371
726	527076	26.9444	.001377	.0371
727	528529	26.9629	.001376	.0371
728	529984	26.9815	.001374	.0371
729	531441	27.0000	.001372	.0370
730	532900	27.0185	.001370	.0370
731	534361	27.0370	.001368	.0370
732	535824	27.0555	.001366	.0370
733	537289	27.0740	.001364	.0369
734	538756	27.0924	.001362	.0369
735	540225	27.1109	.001361	.0369

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
736	541696	27.1293	.001359	.0369
737	543169	27.1477	.001357	.0368
738	544644	27.1662	.001355	.0368
739	546121	27.1846	.001353	.0368
740	547600	27.2029	.001351	.0368
741	549081	27.2213	.001350	.0367
742	550564	27.2397	.001348	.0367
743	552049	27.2580	.001346	.0367
744	553536	27.2764	.001344	.0367
745	555025	27.2947	.001342	.0366
746	556516	27.3130	.001340	.0366
747	558009	27.3313	.001339	.0366
748	559504	27.3496	.001337	.0366
749	561001	27.3679	.001335	.0365
750	562500	27.3861	.001333	.0365
751	564001	27.4044	.001332	.0365
752	565504	27.4226	.001330	.0365
753	567009	27.4408	.001328	.0364
754	568516	27.4591	.001326	.0364
755	570025	27.4773	.001325	.0364
756	571536	27.4955	.001323	.0364
757	573049	27.5136	.001321	.0363
758	574564	27.5318	.001319	.0363
759	576081	27.5500	.001318	.0363
760	577600	27.5681	.001316	.0363
761	579121	27.5862	.001314	.0363
762	580644	27.6043	.001312	.0362
763	582169	27.6225	.001311	.0362
764	583696	27.6405	.001309	.0362
765	585225	27.6586	.001307	.0362
766	586756	27.6767	.001305	.0361
767	588289	27.6948	.001304	.0361
768	589824	27.7128	.001302	.0361
769	591361	27.7308	.001300	.0361
770	592900	27.7489	.001299	.0360
771	594441	27.7669	.001297	.0360
772	595984	27.7849	.001295	.0360
773	597529	27.8029	.001294	.0360
774	599076	27.8209	.001292	.0359
775	600625	27.8388	.001290	.0359
776	602176	27.8568	.001289	.0359
777	603729	27.8747	.001287	.0359
778	605284	27.8927	.001285	.0359
779	606841	27.9106	.001284	.0358
780	608400	27.9285	.001282	.0358
781	609961	27.9464	.001280	.0358
782	611524	27.9643	.001279	.0358
783	613089	27.9821	.001277	.0357
784	614656	28.0000	.001276	.0357
785	616225	28.0179	.001274	.0357

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
786	617796	28.0357	.001272	.0357
787	619369	28.0535	.001271	.0356
788	620944	28.0713	.001269	.0356
789	622521	28.0891	.001267	.0356
790	624100	28.1069	.001266	.0356
791	625681	28.1247	.001264	.0356
792	627264	28.1425	.001263	.0355
793	628849	28.1603	.001261	.0355
794	630436	28.1780	.001259	.0355
795	632025	28.1957	.001258	.0355
796	633616	28.2135	.001256	.0354
797	635209	28.2312	.001255	.0354
798	636804	28.2489	.001253	.0354
799	638401	28.2666	.001252	.0354
800	640000	28.2843	.001250	.0354
801	641601	28.3019	.001248	.0353
802	643204	28.3196	.001247	.0353
803	644809	28.3373	.001245	.0353
804	646416	28.3549	.001244	.0353
805	648025	28.3725	.001242	.0352
806	649636	28.3901	.001241	.0352
807	651249	28.4077	.001239	.0352
808	652864	28.4253	.001238	.0352
809	654481	28.4429	.001236	.0352
810	656100	28.4605	.001235	.0351
811	657721	28.4781	.001233	.0351
812	659344	28.4956	.001232	.0351
813	660969	28.5132	.001230	.0351
814	662596	28.5307	.001229	.0351
815	664225	28.5482	.001227	.0350
816	665856	28.5657	.001225	.0350
817	667489	28.5832	.001224	.0350
818	669124	28.6007	.001222	.0350
819	670761	28.6182	.001221	.0349
820	672400	28.6356	.001220	.0349
821	674041	28.6531	.001218	.0349
822	675684	28.6705	.001217	.0349
823	677329	28.6880	.001215	.0349
824	678976	28.7054	.001214	.0348
825	680625	28.7228	.001212	.0348
826	682276	28.7402	.001211	.0348
827	683929	28.7576	.001209	.0348
828	685584	28.7750	.001208	.0348
829	687241	28.7924	.001206	.0347
830	688900	28.8097	.001205	.0347
831	690561	28.8271	.001203	.0347
832	692224	28.8444	.001202	.0347
833	693889	28.8617	.001200	.0346
834	695556	28.8791	.001199	.0346
835	697225	28.8964	.001198	.0346

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
836	698896	28.9137	.001196	.0346
837	700569	28.9310	.001195	.0346
838	702244	28.9482	.001193	.0345
839	703921	28.9655	.001192	.0345
840	705600	28.9828	.001190	.0345
841	707281	29.0000	.001189	.0345
842	708964	29.0172	.001188	.0345
843	710649	29.0345	.001186	.0344
844	712336	29.0517	.001185	.0344
845	714025	29.0689	.001183	.0344
846	715716	29.0861	.001182	.0344
847	717409	29.1033	.001181	.0344
848	719104	29.1204	.001179	.0343
849	720801	29.1376	.001178	.0343
850	722500	29.1548	.001176	.0343
851	724201	29.1719	.001175	.0343
852	725904	29.1890	.001174	.0343
853	727609	29.2062	.001172	.0342
854	729316	29.2233	.001171	.0342
855	731025	29.2404	.001170	.0342
856	732736	29.2575	.001168	.0342
857	734449	29.2746	.001167	.0342
858	736164	29.2916	.001166	.0341
859	737881	29.3087	.001164	.0341
860	739600	29.3258	.001163	.0341
861	741321	29.3428	.001161	.0341
862	743044	29.3598	.001160	.0341
863	744769	29.3769	.001159	.0340
864	746496	29.3939	.001157	.0340
865	748225	29.4109	.001156	.0340
866	749956	29.4279	.001155	.0340
867	751689	29.4449	.001153	.0340
868	753424	29.4618	.001152	.0339
869	755161	29.4788	.001151	.0339
870	756900	29.4958	.001149	.0339
871	758641	29.5127	.001148	.0339
872	760384	29.5296	.001147	.0339
873	762129	29.5466	.001145	.0338
874	763876	29.5635	.001144	.0338
875	765625	29.5804	.001143	.0338
876	767376	29.5973	.001142	.0338
877	769129	29.6142	.001140	.0338
878	770884	29.6311	.001139	.0337
879	772641	29.6479	.001138	.0337
880	774400	29.6648	.001136	.0337
881	776161	29.6816	.001135	.0337
882	777924	29.6985	.001134	.0337
883	779689	29.7153	.001133	.0337
884	781456	29.7321	.001131	.0336
885	783225	29.7489	.001130	.0336

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
886	784996	29.7658	.001129	.0336
887	786769	29.7825	.001127	.0336
888	788544	29.7993	.001126	.0336
889	790321	29.8161	.001125	.0335
890	792100	29.8329	.001124	.0335
891	793881	29.8496	.001122	.0335
892	795664	29.8664	.001121	.0335
893	797449	29.8831	.001120	.0335
894	799236	29.8998	.001119	.0334
895	801025	29.9166	.001117	.0334
896	802816	29.9333	.001116	.0334
897	804609	29.9500	.001115	.0334
898	806404	29.9666	.001114	.0334
899	808201	29.9833	.001112	.0334
900	810000	30.0000	.001111	.0333
901	811801	30.0167	.001110	.0333
902	813604	30.0333	.001109	.0333
903	815409	30.0500	.001107	.0333
904	817216	30.0666	.001106	.0333
905	819025	30.0832	.001105	.0332
906	820836	30.0998	.001104	.0332
907	822649	30.1164	.001103	.0332
908	824464	30.1330	.001101	.0332
909	826281	30.1496	.001100	.0332
910	828100	30.1662	.001099	.0331
911	829921	30.1828	.001098	.0331
912	831744	30.1993	.001096	.0331
913	833569	30.2159	.001095	.0331
914	835396	30.2324	.001094	.0331
915	837225	30.2490	.001093	.0331
916	839056	30.2655	.001092	.0330
917	840889	30.2820	.001091	.0330
918	842724	30.2985	.001089	.0330
919	844561	30.3150	.001088	.0330
920	846400	30.3315	.001087	.0330
921	848241	30.3480	.001086	.0330
922	850084	30.3645	.001085	.0329
923	851929	30.3809	.001083	.0329
924	853776	30.3974	.001082	.0329
925	855625	30.4138	.001081	.0329
926	857476	30.4302	.001080	.0329
927	859329	30.4467	.001079	.0328
928	861184	30.4631	.001078	.0328
929	863041	30.4795	.001076	.0328
930	864900	30.4959	.001075	.0328
931	866761	30.5123	.001074	.0328
932	868624	30.5287	.001073	.0328
933	870489	30.5450	.001072	.0327
934	872356	30.5614	.001071	.0327
935	874225	30.5778	.001070	.0327

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
936	876096	30.5941	.001068	.0327
937	877969	30.6105	.001067	.0327
938	879844	30.6268	.001066	.0327
939	881721	30.6431	.001065	.0326
940	883600	30.6594	.001064	.0326
941	885481	30.6757	.001063	.0326
942	887364	30.6920	.001062	.0326
943	889249	30.7083	.001060	.0326
944	891136	30.7246	.001059	.0325
945	893025	30.7409	.001058	.0325
946	894916	30.7571	.001057	.0325
947	896809	30.7734	.001056	.0325
948	898704	30.7896	.001055	.0325
949	900601	30.8058	.001054	.0325
950	902500	30.8221	.001053	.0324
951	904401	30.8383	.001052	.0324
952	906304	30.8545	.001050	.0324
953	908209	30.8707	.001049	.0324
954	910116	30.8869	.001048	.0324
955	912025	30.9031	.001047	.0324
956	913936	30.9192	.001046	.0323
957	915849	30.9354	.001045	.0323
958	917764	30.9516	.001044	.0323
959	919681	30.9677	.001043	.0323
960	921600	30.9839	.001042	.0323
961	923521	31.0000	.001041	.0323
962	925444	31.0161	.001040	.0322
963	927369	31.0322	.001038	.0322
964	929296	31.0483	.001037	.0322
965	931225	31.0644	.001036	.0322
966	933156	31.0805	.001035	.0322
967	935089	31.0966	.001034	.0322
968	937024	31.1127	.001033	.0321
969	938961	31.1288	.001032	.0321
970	940900	31.1448	.001031	.0321
971	942841	31.1609	.001030	.0321
972	944784	31.1769	.001029	.0321
973	946729	31.1929	.001028	.0321
974	948676	31.2090	.001027	.0320
975	950625	31.2250	.001026	.0320
976	952576	31.2410	.001025	.0320
977	954529	31.2570	.001024	.0320
978	956484	31.2730	.001022	.0320
979	958441	31.2890	.001021	.0320
980	960400	31.3050	.001020	.0319
981	962361	31.3209	.001019	.0319
982	964324	31.3369	.001018	.0319
983	966289	31.3528	.001017	.0319
984	968256	31.3688	.001016	.0319
985	970225	31.3847	.001015	.0319

TABLE I (*Continued*)

n	n^2	$\sqrt{n}$	$\frac{1}{n}$	$\frac{1}{\sqrt{n}}$
986	972196	31.4006	.001014	.0318
987	974169	31.4166	.001013	.0318
988	976144	31.4325	.001012	.0318
989	978121	31.4484	.001011	.0318
990	980100	31.4643	.001010	.0318
991	982081	31.4802	.001009	.0318
992	984064	31.4960	.001008	.0318
993	986049	31.5119	.001007	.0317
994	988036	31.5278	.001006	.0317
995	990025	31.5436	.001005	.0317
996	992016	31.5595	.001004	.0317
997	994009	31.5753	.001003	.0317
998	996004	31.5911	.001002	.0317
999	998001	31.6070	.001001	.0316
1000	1000000	31.6228	.001000	.0316

TABLE II. Areas and Ordinates of the Normal Curve in Terms of x/σ

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
0.00	.0000	.5000	.5000	.3989
0.01	.0040	.5040	.4960	.3989
0.02	.0080	.5080	.4920	.3989
0.03	.0120	.5120	.4880	.3988
0.04	.0160	.5160	.4840	.3986
0.05	.0199	.5199	.4801	.3984
0.06	.0239	.5239	.4761	.3982
0.07	.0279	.5279	.4721	.3980
0.08	.0319	.5319	.4681	.3977
0.09	.0359	.5359	.4641	.3973
0.10	.0398	.5398	.4602	.3970
0.11	.0438	.5438	.4562	.3965
0.12	.0478	.5478	.4522	.3961
0.13	.0517	.5517	.4483	.3956
0.14	.0557	.5557	.4443	.3951
0.15	.0596	.5596	.4404	.3945
0.16	.0636	.5636	.4364	.3939
0.17	.0675	.5675	.4325	.3932
0.18	.0714	.5714	.4286	.3925
0.19	.0753	.5753	.4247	.3918
0.20	.0793	.5793	.4207	.3910
0.21	.0832	.5832	.4168	.3902
0.22	.0871	.5871	.4129	.3894
0.23	.0910	.5910	.4090	.3885
0.24	.0948	.5948	.4052	.3876
0.25	.0987	.5987	.4013	.3867
0.26	.1026	.6026	.3974	.3857
0.27	.1064	.6064	.3936	.3847
0.28	.1103	.6103	.3897	.3836
0.29	.1141	.6141	.3859	.3825
0.30	.1179	.6179	.3821	.3814
0.31	.1217	.6217	.3783	.3802
0.32	.1255	.6255	.3745	.3790
0.33	.1293	.6293	.3707	.3778
0.34	.1331	.6331	.3669	.3765
0.35	.1368	.6368	.3632	.3752
0.36	.1406	.6406	.3594	.3739
0.37	.1443	.6443	.3557	.3725
0.38	.1480	.6480	.3520	.3712
0.39	.1517	.6517	.3483	.3697
0.40	.1554	.6554	.3446	.3683
0.41	.1591	.6591	.3409	.3668
0.42	.1628	.6628	.3372	.3653
0.43	.1664	.6664	.3336	.3637
0.44	.1700	.6700	.3300	.3621

SOURCE: A. L. Edwards. *Statistical Methods for the Behavioral Sciences.* New York: Rinehart, 1954. Reprinted by permission of the publisher.

TABLE II (*Continued*)

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
0.45	.1736	.6736	.3264	.3605
0.46	.1772	.6772	.3228	.3589
0.47	.1808	.6808	.3192	.3572
0.48	.1844	.6844	.3156	.3555
0.49	.1879	.6879	.3121	.3538
0.50	.1915	.6915	.3085	.3521
0.51	.1950	.6950	.3050	.3503
0.52	.1985	.6985	.3015	.3485
0.53	.2019	.7019	.2981	.3467
0.54	.2054	.7054	.2946	.3448
0.55	.2088	.7088	.2912	.3429
0.56	.2123	.7123	.2877	.3410
0.57	.2157	.7157	.2843	.3391
0.58	.2190	.7190	.2810	.3372
0.59	.2224	.7224	.2776	.3352
0.60	.2257	.7257	.2743	.3332
0.61	.2291	.7291	.2709	.3312
0.62	.2324	.7324	.2676	.3292
0.63	.2357	.7357	.2643	.3271
0.64	.2389	.7389	.2611	.3251
0.65	.2422	.7422	.2578	.3230
0.66	.2454	.7454	.2546	.3209
0.67	.2486	.7486	.2514	.3187
0.68	.2517	.7517	.2483	.3166
0.69	.2549	.7549	.2451	.3144
0.70	.2580	.7580	.2420	.3123
0.71	.2611	.7611	.2389	.3101
0.72	.2642	.7642	.2358	.3079
0.73	.2673	.7673	.2327	.3056
0.74	.2704	.7704	.2296	.3034
0.75	.2734	.7734	.2266	.3011
0.76	.2764	.7764	.2236	.2989
0.77	.2794	.7794	.2206	.2966
0.78	.2823	.7823	.2177	.2943
0.79	.2852	.7852	.2148	.2920
0.80	.2881	.7881	.2119	.2897
0.81	.2910	.7910	.2090	.2874
0.82	.2939	.7939	.2061	.2850
0.83	.2967	.7967	.2033	.2827
0.84	.2995	.7995	.2005	.2803
0.85	.3023	.8023	.1977	.2780
0.86	.3051	.8051	.1949	.2756
0.87	.3078	.8078	.1922	.2732
0.88	.3106	.8106	.1894	.2709
0.89	.3133	.8133	.1867	.2685

TABLE II (*Continued*)

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
0.90	.3159	.8159	.1841	.2661
0.91	.3186	.8186	.1814	.2637
0.92	.3212	.8212	.1788	.2613
0.93	.3238	.8238	.1762	.2589
0.94	.3264	.8264	.1736	.2565
0.95	.3289	.8289	.1711	.2541
0.96	.3315	.8315	.1685	.2516
0.97	.3340	.8340	.1660	.2492
0.98	.3365	.8365	.1635	.2468
0.99	.3389	.8389	.1611	.2444
1.00	.3413	.8413	.1587	.2420
1.01	.3438	.8438	.1562	.2396
1.02	.3461	.8461	.1539	.2371
1.03	.3485	.8485	.1515	.2347
1.04	.3508	.8508	.1492	.2323
1.05	.3531	.8531	.1469	.2299
1.06	.3554	.8554	.1446	.2275
1.07	.3577	.8577	.1423	.2251
1.08	.3599	.8599	.1401	.2227
1.09	.3621	.8621	.1379	.2203
1.10	.3643	.8643	.1357	.2179
1.11	.3665	.8665	.1335	.2155
1.12	.3686	.8686	.1314	.2131
1.13	.3708	.8708	.1292	.2107
1.14	.3729	.8729	.1271	.2083
1.15	.3749	.8749	.1251	.2059
1.16	.3770	.8770	.1230	.2036
1.17	.3790	.8790	.1210	.2012
1.18	.3810	.8810	.1190	.1989
1.19	.3830	.8830	.1170	.1965
1.20	.3849	.8849	.1151	.1942
1.21	.3869	.8869	.1131	.1919
1.22	.3888	.8888	.1112	.1895
1.23	.3907	.8907	.1093	.1872
1.24	.3925	.8925	.1075	.1849
1.25	.3944	.8944	.1056	.1826
1.26	.3962	.8962	.1038	.1804
1.27	.3980	.8980	.1020	.1781
1.28	.3997	.8997	.1003	.1758
1.29	.4015	.9015	.0985	.1736
1.30	.4032	.9032	.0968	.1714
1.31	.4049	.9049	.0951	.1691
1.32	.4066	.9066	.0934	.1669
1.33	.4082	.9082	.0918	.1647
1.34	.4099	.9099	.0901	.1626

TABLE II (*Continued*)

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
1.35	.4115	.9115	.0885	.1604
1.36	.4131	.9131	.0869	.1582
1.37	.4147	.9147	.0853	.1561
1.38	.4162	.9162	.0838	.1539
1.39	.4177	.9177	.0823	.1518
1.40	.4192	.9192	.0808	.1497
1.41	.4207	.9207	.0793	.1476
1.42	.4222	.9222	.0778	.1456
1.43	.4236	.9236	.0764	.1435
1.44	.4251	.9251	.0749	.1415
1.45	.4265	.9265	.0735	.1394
1.46	.4279	.9279	.0721	.1374
1.47	.4292	.9292	.0708	.1354
1.48	.4306	.9306	.0694	.1334
1.49	.4319	.9319	.0681	.1315
1.50	.4332	.9332	.0668	.1295
1.51	.4345	.9345	.0655	.1276
1.52	.4357	.9357	.0643	.1257
1.53	.4370	9370	.0630	.1238
1.54	.4382	.9382	.0618	.1219
1.55	.4394	.9394	.0606	.1200
1.56	.4406	.9406	.0594	.1182
1.57	.4418	.9418	.0582	.1163
1.58	.4429	.9429	.0571	.1145
1.59	.4441	.9441	.0559	.1127
1.60	.4452	.9452	.0548	.1109
1.61	.4463	.9463	.0537	.1092
1.62	.4474	.9474	.0526	.1074
1.63	.4484	.9484	.0516	.1057
1.64	.4495	.9495	.0505	.1040
1.65	.4505	.9505	.0495	.1023
1.66	.4515	.9515	.0485	.1006
1.67	.4525	.9525	.0475	.0989
1.68	.4535	.9535	.0465	.0973
1.69	.4545	.9545	.0455	.0957
1.70	.4554	.9554	.0446	.0940
1.71	.4564	.9564	.0436	.0925
1.72	.4573	.9573	.0427	.0909
1.73	.4582	.9582	.0418	.0893
1.74	.4591	.9591	.0409	.0878
1.75	.4599	.9599	.0401	.0863
1.76	.4608	.9608	.0392	.0848
1.77	.4616	.9616	.0384	.0833
1.78	.4625	.9625	.0375	.0818
1.79	.4633	.9633	.0367	.0804

TABLE II (*Continued*)

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
1.80	.4641	.9641	.0359	.0790
1.81	.4649	.9649	.0351	.0775
1.82	.4656	.9656	.0344	.0761
1.83	.4664	.9664	.0336	.0748
1.84	.4671	.9671	.0329	.0734
1.85	.4648	.9678	.0322	.0721
1.86	.4686	.9686	.0314	.0707
1.87	.4693	.9693	.0307	.0694
1.88	.4699	.9699	.0301	.0681
1.89	.4706	.9706	.0294	.0669
1.90	.4713	.9713	.0287	.0656
1.91	.4719	.9719	.0281	.0644
1.92	.4726	.9726	.0274	.0632
1.93	.4732	.9732	.0268	.0620
1.94	.4738	.9738	.0262	.0608
1.95	.4744	.9744	.0256	.0596
1.96	.4750	.9750	.0250	.0584
1.97	.4756	.9756	.0244	.0573
1.98	.4761	.9761	.0239	.0562
1.99	.4767	.9767	.0233	.0551
2.00	.4772	.9772	.0228	.0540
2.01	.4778	.9778	.0222	.0529
2.02	.4783	.9783	.0217	.0519
2.03	.4788	.9788	.0212	.0508
2.04	.4793	.9793	.0207	.0498
2.05	.4798	.9798	.0202	.0488
2.06	.4803	.9803	.0197	.0478
2.07	.4808	.9808	.0192	.0468
2.08	.4812	.9812	.0188	.0459
2.09	.4817	.9817	.0183	.0449
2.10	.4821	.9821	.0179	.0440
2.11	.4826	.9826	.0174	.0431
2.12	.4830	.9830	.0170	.0422
2.13	.4834	.9834	.0166	.0413
2.14	.4838	.9838	.0162	.0404
2.15	.4842	.9842	.0158	.0396
2.16	.4846	.9846	.0154	.0387
2.17	.4850	.9850	.0150	.0379
2.18	.4854	.9854	.0146	.0371
2.19	.4857	.9857	.0143	.0363
2.20	.4861	.9861	.0139	.0355
2.21	.4864	.9864	.0136	.0347
2.22	.4868	.9868	.0132	.0339
2.23	.4871	.9871	.0129	.0332
2.24	.4875	.9875	.0125	.0325

TABLE II (*Continued*)

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
2.25	.4878	.9878	.0122	.0317
2.26	.4881	.9881	.0119	.0310
2.27	.4884	.9884	.0116	.0303
2.28	.4887	.9887	.0113	.0297
2.29	.4890	.9890	.0110	.0290
2.30	.4893	.9893	.0107	.0283
2.31	.4896	.9896	.0104	.0277
2.32	.4898	.9898	.0102	.0270
2.33	.4901	.9901	.0099	.0264
2.34	.4904	.9904	.0096	.0258
2.35	.4906	.9906	.0094	.0252
2.36	.4909	.9909	.0091	.0246
2.37	.4911	.9911	.0089	.0241
2.38	.4913	.9913	.0087	.0235
2.39	.4916	.9916	.0084	.0229
2.40	.4918	.9918	.0082	.0224
2.41	.4920	.9920	.0080	.0219
2.42	.4922	.9922	.0078	.0213
2.43	.4925	.9925	.0075	.0208
2.44	.4927	.9927	.0073	.0203
2.45	.4929	.9929	.0071	.0198
2.46	.4931	.9931	.0069	.0194
2.47	.4932	.9932	.0068	.0189
2.48	.4934	.9934	.0066	.0184
2.49	.4936	.9936	.0064	.0180
2.50	.4938	.9938	.0062	.0175
2.51	.4940	.9940	.0060	.0171
2.52	.4941	.9941	.0059	.0167
2.53	.4943	.9943	.0057	.0163
2.54	.4945	.9945	.0055	.0158
2.55	.4946	.9946	.0054	.0154
2.56	.4948	.9948	.0052	.0151
2.57	.4949	.9949	.0051	.0147
2.58	.4951	.9951	.0049	.0143
2.59	.4952	.9952	.0048	.0139
2.60	.4953	.9953	.0047	.0136
2.61	.4955	.9955	.0045	.0132
2.62	.4956	.9956	.0044	.0129
2.63	.4957	.9957	.0043	.0126
2.64	.4959	.9959	.0041	.0122
2.65	.4960	.9960	.0040	.0119
2.66	.4961	.9961	.0039	.0116
2.67	.4962	.9962	.0038	.0113
2.68	.4963	.9963	.0037	.0110
2.69	.4964	.9964	.0036	.0107

TABLE II (*Continued*)

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
2.70	.4965	.9965	.0035	.0104
2.71	.4966	.9966	.0034	.0101
2.72	.4967	.9967	.0033	.0099
2.73	.4968	.9968	.0032	.0096
2.74	.4969	.9969	.0031	.0093
2.75	.4970	.9970	.0030	.0091
2.76	.4971	.9971	.0029	.0088
2.77	.4972	.9972	.0028	.0086
2.78	.4973	.9973	.0027	.0084
2.79	.4974	.9974	.0026	.0081
2.80	.4974	.9974	.0026	.0079
2.81	.4975	.9975	.0025	.0077
2.82	.4976	.9976	.0024	.0075
2.83	.4977	.9977	.0023	.0073
2.84	.4977	.9977	.0023	.0071
2.85	.4978	.9978	.0022	.0069
2.86	.4979	.9979	.0021	.0067
2.87	.4979	.9979	.0021	.0065
2.88	.4980	.9980	.0020	.0063
2.89	.4981	.9981	.0019	.0061
2.90	.4981	.9981	.0019	.0060
2.91	.4982	.9982	.0018	.0058
2.92	.4982	.9982	.0018	.0056
2.93	.4983	.9983	.0017	.0055
2.94	.4984	.9984	.0016	.0053
2.95	.4984	.9984	.0016	.0051
2.96	.4985	.9985	.0015	.0050
2.97	.4985	.9985	.0015	.0048
2.98	.4986	.9986	.0014	.0047
2.99	.4986	.9986	.0014	.0046
3.00	.4987	.9987	.0013	.0044
3.01	.4987	.9987	.0013	.0043
3.02	.4987	.9987	.0013	.0042
3.03	.4988	.9988	.0012	.0040
3.04	.4988	.9988	.0012	.0039
3.05	.4989	.9989	.0011	.0038
3.06	.4989	.9989	.0011	.0037
3.07	.4989	.9989	.0011	.0036
3.08	.4990	.9990	.0010	.0035
3.09	.4990	.9990	.0010	.0034
3.10	.4990	.9990	.0010	.0033
3.11	.4991	.9991	.0009	.0032
3.12	.4991	.9991	.0009	.0031
3.13	.4991	.9991	.0009	.0030
3.14	.4992	.9992	.0008	.0029

TABLE II (*Continued*)

(1) z Standard Score $\left(\frac{x}{\sigma}\right)$	(2) A Area from Mean to $\frac{x}{\sigma}$	(3) B Area in Larger Portion	(4) C Area in Smaller Portion	(5) y Ordinate at $\frac{x}{\sigma}$
3.15	.4992	.9992	.0008	.0028
3.16	.4992	.9992	.0008	.0027
3.17	.4992	.9992	.0008	.0026
3.18	.4993	.9993	.0007	.0025
3.19	.4993	.9993	.0007	.0025
3.20	.4993	.9993	.0007	.0024
3.21	.4993	.9993	.0007	.0023
3.22	.4994	.9994	.0006	.0022
3.23	.4994	.9994	.0006	.0022
3.24	.4994	.9994	.0006	.0021
3.30	.4995	.9995	.0005	.0017
3.40	.4997	.9997	.0003	.0012
3.50	.4998	.9998	.0002	.0009
3.60	.4998	.9998	.0002	.0006
3.70	.4999	.9999	.0001	.0004

TABLE III. Distribution of t Probability

n	.1	.05	.01	.001
1	6.314	12.706	63.657	636.619
2	2.920	4.303	9.925	31.598
3	2.353	3.182	5.841	12.941
4	2.132	2.776	4.604	8.610
5	2.015	2.571	4.032	6.859
6	1.943	2.447	3.707	5.959
7	1.895	2.365	3.499	5.405
8	1.860	2.306	3.355	5.041
9	1.833	2.262	3.250	4.781
10	1.812	2.228	3.169	4.587
11	1.796	2.201	3.106	4.437
12	1.782	2.179	3.055	4.318
13	1.771	2.160	3.012	4.221
14	1.761	2.145	2.977	4.140
15	1.753	2.131	2.947	4.073
16	1.746	2.120	2.921	4.015
17	1.740	2.110	2.898	3.965
18	1.734	2.101	2.878	3.922
19	1.729	2.093	2.861	3.883
20	1.725	2.086	2.845	3.850
21	1.721	2.080	2.831	3.819
22	1.717	2.074	2.819	3.792
23	1.714	2.069	2.807	3.767
24	1.711	2.064	2.797	3.745
25	1.708	2.060	2.787	3.725
26	1.706	2.056	2.779	3.707
27	1.703	2.052	2.771	3.690
28	1.701	2.048	2.763	3.674
29	1.699	2.045	2.756	3.659
30	1.697	2.042	2.750	3.646
40	1.684	2.021	2.704	3.551
60	1.671	2.000	2.660	3.460
120	1.658	1.980	2.617	3.373
∞	1.645	1.960	2.576	3.291

SOURCE: Table III is abridged from Table III of R. A. Fisher and F. Yates: *Statistical Tables for Biological, Agricultural, and Medical Research*, published by Oliver and Boyd Ltd., Edinburgh, by permission of the authors and publisher.

TABLE IV. Distribution of χ^2

	Probability													
df	.99	.98	.95	.90	.80	.70	.50	.30	.20	.10	.05	.02	.01	.001
1	$.0^{3}157$	$.0^{3}628$	.00393	.0158	.0642	.148	.455	1.074	1.642	2.706	3.841	5.412	6.635	10.827
2	.0201	.0404	.103	.211	.446	.713	1.386	2.408	3.219	4.605	5.991	7.824	9.210	13.815
3	.115	.185	.352	.584	1.005	1.424	2.366	3.665	4.642	6.251	7.815	9.837	11.345	16.268
4	.297	.429	.711	1.064	1.649	2.195	3.357	4.878	5.989	7.779	9.488	11.668	13.277	18.465
5	.554	.752	1.145	1.610	2.343	3.000	4.351	6.064	7.289	9.236	11.070	13.388	15.086	20.517
6	.872	1.134	1.635	2.204	3.070	3.828	5.348	7.231	8.558	10.645	12.592	15.033	16.812	22.457
7	1.239	1.564	2.167	2.833	3.822	4.671	6.346	8.383	9.803	12.017	14.067	16.622	18.475	24.322
8	1.646	2.032	2.733	3.490	4.594	5.527	7.344	9.524	11.030	13.362	15.507	18.168	20.090	26.125
9	2.088	2.532	3.325	4.168	5.380	6.393	8.343	10.656	12.242	14.684	16.919	19.679	21.666	27.877
10	2.558	3.059	3.940	4.865	6.179	7.267	9.342	11.781	13.442	15.987	18.307	21.161	23.209	29.588
11	3.053	3.609	4.575	5.578	6.989	8.148	10.341	12.899	14.631	17.275	19.675	22.618	24.725	31.264
12	3.571	4.178	5.226	6.304	7.807	9.034	11.340	14.011	15.812	18.549	21.026	24.054	26.217	32.909
13	4.107	4.765	5.892	7.042	8.634	9.926	12.340	15.119	16.985	19.812	22.362	25.472	27.688	34.528
14	4.660	5.368	6.571	7.790	9.467	10.821	13.339	16.222	18.151	21.064	23.685	26.873	29.141	36.123
15	5.229	5.985	7.261	8.547	10.307	11.721	14.339	17.322	19.311	22.307	24.996	28.259	30.578	37.697
16	5.812	6.614	7.962	9.312	11.152	12.624	15.338	18.418	20.465	23.542	26.296	29.633	32.000	39.252
17	6.408	7.255	8.672	10.085	12.002	13.531	16.338	19.511	21.615	24.769	27.587	30.995	33.409	40.790
18	7.015	7.906	9.390	10.865	12.857	14.440	17.338	20.601	22.760	25.989	28.869	32.346	34.805	42.312
19	7.633	8.567	10.117	11.651	13.716	15.352	18.338	21.689	23.900	27.204	30.144	33.687	36.191	43.820
20	8.260	9.237	10.851	12.443	14.578	16.266	19.337	22.775	25.038	28.412	31.410	35.020	37.566	45.315
21	8.897	9.915	11.591	13.240	15.445	17.182	20.337	23.858	26.171	29.615	32.671	36.343	38.932	46.797
22	9.542	10.600	12.338	14.041	16.314	18.101	21.337	24.939	27.301	30.813	33.924	37.659	40.289	48.268
23	10.196	11.293	13.091	14.848	17.187	19.021	22.337	26.018	28.429	32.007	35.172	38.968	41.638	49.728
24	10.856	11.992	13.848	15.659	18.062	19.943	23.337	27.096	29.553	33.196	36.415	40.270	42.980	51.179
25	11.524	12.697	14.611	16.473	18.940	20.867	24.337	28.172	30.675	34.382	37.652	41.566	44.314	52.620
26	12.198	13.409	15.379	17.292	19.820	21.792	25.336	29.246	31.795	35.563	38.885	42.856	45.642	54.052
27	12.879	14.125	16.151	18.114	20.703	22.719	26.336	30.319	32.912	36.741	40.113	44.140	46.963	55.476
28	13.565	14.847	16.928	18.939	21.588	23.647	27.336	31.391	34.027	37.916	41.337	45.419	48.278	56.893
29	14.256	15.574	17.708	19.768	22.475	24.577	28.336	32.461	35.139	39.087	42.557	46.693	49.588	58.302
30	14.953	16.306	18.493	20.599	23.364	25.508	29.336	33.530	36.250	40.256	43.773	47.962	50.892	59.703

SOURCE: Table IV is reprinted from Table IV of R. A. Fisher and F. Yates: *Statistical Tables for Biological, Agricultural, and Medical Research*, published by Oliver and Boyd Ltd., Edinburgh, by permission of the authors and publishers.

TABLE V. 5 Percent (Lightface Type) and 1 Percent

									f_1 Degrees of Freedom (for			
f_2	1	2	3	4	5	6	7	8	9	10	11	12
1	161 **4,052**	200 **4,999**	216 **5,403**	225 **5,625**	230 **5,764**	234 **5,859**	237 **5,928**	239 **5,981**	241 **6,022**	242 **6,056**	243 **6,082**	244 **6,106**
2	18.51 **98.49**	19.00 **99.00**	19.16 **99.17**	19.25 **99.25**	19.30 **99.30**	19.33 **99.33**	19.36 **99.34**	19.37 **99.36**	19.38 **99.38**	19.39 **99.40**	19.40 **99.41**	19.41 **99.42**
3	10.13 **34.12**	9.55 **30.82**	9.28 **29.46**	9.12 **28.71**	9.01 **28.24**	8.94 **27.91**	8.88 **29.67**	8.84 **27.49**	8.81 **27.34**	8.78 **27.23**	8.76 **27.13**	8.74 **27.05**
4	7.71 **21.20**	6.94 **18.00**	6.59 **16.69**	6.39 **15.98**	6.26 **15.52**	6.16 **15.21**	6.09 **14.98**	6.04 **14.80**	6.00 **14.66**	5.96 **14.54**	5.93 **14.45**	5.91 **14.37**
5	6.61 **16.26**	5.79 **13.27**	5.41 **12.06**	5.19 **11.39**	5.05 **10.97**	4.95 **10.67**	4.88 **10.45**	4.82 **10.27**	4.78 **10.15**	4.74 **10.05**	4.70 **9.96**	4.68 **9.89**
6	5.99 **13.74**	5.14 **10.92**	4.76 **9.78**	4.53 **9.15**	4.39 **8.75**	4.28 **8.47**	4.21 **8.26**	4.15 **8.10**	4.10 **7.98**	4.06 **7.87**	4.03 **7.79**	4.00 **7.72**
7	5.59 **12.25**	4.74 **9.55**	4.35 **8.45**	4.12 **7.85**	3.97 **7.46**	3.87 **7.19**	3.79 **7.00**	3.73 **6.84**	3.68 **6.71**	3.63 **6.62**	3.60 **6.54**	3.57 **6.47**
8	5.32 **11.26**	4.46 **8.65**	4.07 **7.59**	3.84 **7.01**	3.69 **6.63**	3.58 **6.37**	3.50 **6.19**	3.44 **6.03**	3.39 **5.91**	3.34 **5.82**	3.31 **5.74**	3.28 **5.67**
9	5.12 **10.56**	4.26 **8.02**	3.86 **6.99**	3.63 **6.42**	3.48 **6.06**	3.37 **5.80**	3.29 **5.62**	3.23 **5.47**	3.18 **5.35**	3.13 **5.26**	3.10 **5.18**	3.07 **5.11**
10	4.96 **10.04**	4.10 **7.56**	3.71 **6.55**	3.48 **5.99**	3.33 **5.64**	3.22 **5.39**	3.14 **5.21**	3.07 **5.06**	3.02 **4.95**	2.97 **4.85**	2.94 **4.78**	2.91 **4.71**
11	4.84 **9.65**	3.98 **7.20**	3.59 **6.22**	3.36 **5.67**	3.20 **5.32**	3.09 **5.07**	3.01 **4.88**	2.95 **4.74**	2.90 **4.63**	2.86 **4.54**	2.82 **4.46**	2.79 **4.40**
12	4.75 **9.33**	3.88 **6.93**	3.49 **5.95**	3.26 **5.41**	3.11 **5.06**	3.00 **4.82**	2.92 **4.65**	2.85 **4.50**	2.80 **4.39**	2.76 **4.30**	2.72 **4.22**	2.69 **4.16**
13	4.67 **9.07**	3.80 **6.70**	3.41 **5.74**	3.18 **5.20**	3.02 **4.86**	2.92 **4.62**	2.84 **4.44**	2.72 **4.30**	2.77 **4.19**	2.63 **4.10**	2.63 **4.02**	2.60 **3.96**
14	4.60 **8.86**	3.74 **6.51**	3.34 **5.56**	3.11 **5.03**	2.96 **4.69**	2.85 **4.46**	2.77 **4.28**	2.70 **4.14**	2.65 **4.03**	2.60 **3.94**	2.56 **3.86**	2.53 **3.80**
15	4.54 **8.68**	3.68 **6.36**	3.29 **5.42**	3.06 **4.89**	2.90 **4.56**	2.79 **4.32**	2.70 **4.14**	2.64 **4.00**	2.59 **3.89**	2.55 **3.80**	2.51 **3.73**	2.48 **3.67**
16	4.49 **8.53**	3.63 **6.23**	3.24 **5.29**	3.01 **4.77**	2.85 **4.44**	2.74 **4.20**	2.66 **4.03**	2.59 **3.89**	2.54 **3.78**	2.49 **3.69**	2.45 **3.61**	2.42 **3.55**
17	4.45 **8.40**	3.59 **6.11**	3.20 **5.18**	2.96 **4.67**	2.81 **4.34**	2.70 **4.10**	2.62 **3.93**	2.55 **3.79**	2.50 **3.68**	2.45 **3.59**	2.41 **3.52**	2.38 **3.45**
18	4.41 **8.28**	3.55 **6.01**	3.16 **5.09**	2.93 **4.58**	2.77 **4.25**	2.66 **4.01**	2.58 **3.85**	2.51 **3.71**	2.46 **3.60**	2.41 **3.51**	2.37 **3.44**	2.34 **3.37**

SOURCE: G. W. Snedecor. *Statistical Methods.* Ames, Iowa: Iowa State

(Boldface Type) Points for the Distribution of F

Greater Mean Square)

14	16	20	24	30	40	50	75	100	200	500	∞	f_2
245 **6,142**	246 **6,169**	248 **6,208**	249 **6,234**	250 **6,258**	251 **6,286**	252 **6,302**	253 **6,323**	253 **6,334**	254 **6,352**	254 **6,361**	254 **6,366**	1
19.42 **99.43**	19.43 **99.44**	19.44 **99.45**	19.45 **99.46**	19.46 **99.47**	19.47 **99.48**	19.47 **99.48**	19.48 **99.49**	19.49 **99.49**	19.49 **99.49**	19.50 **99.50**	19.50 **99.50**	2
8.71 **26.92**	8.69 **26.83**	8.66 **26.69**	8.64 **26.60**	8.62 **26.50**	8.60 **26.41**	8.58 **26.35**	8.57 **26.27**	8.56 **26.23**	8.54 **26.18**	8.54 **26.14**	8.53 **26.12**	3
5.87 **14.24**	5.84 **14.15**	5.80 **14.02**	5.77 **13.93**	5.74 **13.83**	5.71 **13.74**	5.70 **13.69**	5.68 **13.61**	5.66 **13.57**	5.65 **13.52**	5.64 **13.48**	5.63 **13.46**	4
4.64 **9.77**	4.60 **9.68**	4.56 **9.55**	4.53 **9.47**	4.50 **9.38**	4.46 **9.29**	4.44 **9.24**	4.42 **9.17**	4.40 **9.13**	4.38 **9.07**	4.37 **9.04**	4.36 **9.02**	5
3.96 **7.60**	3.92 **7.52**	3.87 **7.39**	3.84 **7.31**	3.81 **7.23**	3.77 **7.14**	3.75 **7.09**	3.72 **7.02**	3.71 **6.99**	3.69 **6.94**	3.68 **6.90**	3.67 **6.88**	6
3.52 **6.35**	3.49 **6.27**	3.44 **6.15**	3.41 **6.07**	3.38 **5.98**	3.34 **5.90**	3.32 **5.85**	3.29 **5.78**	3.28 **5.75**	3.25 **5.70**	3.24 **5.67**	3.23 **5.65**	7
3.23 **5.56**	3.20 **5.48**	3.15 **5.36**	3.12 **5.28**	3.08 **5.20**	3.05 **5.11**	3.03 **5.06**	3.00 **5.00**	2.98 **4.96**	2.96 **4.91**	2.94 **4.88**	2.93 **4.86**	8
3.02 **5.00**	2.98 **4.92**	2.93 **4.80**	2.90 **4.73**	2.86 **4.64**	2.82 **4.56**	2.80 **4.51**	2.77 **4.45**	2.76 **4.41**	2.73 **4.36**	2.72 **4.33**	2.71 **4.31**	9
2.86 **4.60**	2.82 **4.52**	2.77 **4.41**	2.74 **4.33**	2.70 **4.25**	2.67 **4.17**	2.64 **4.12**	2.61 **4.05**	2.59 **4.01**	2.56 **3.96**	2.55 **3.93**	2.54 **3.91**	10
2.74 **4.29**	2.70 **4.21**	2.65 **4.10**	2.61 **4.02**	2.57 **3.94**	2.53 **3.86**	2.50 **3.80**	2.47 **3.74**	2.45 **3.70**	2.42 **3.66**	2.41 **3.62**	2.40 **3.60**	11
2.64 **4.05**	2.60 **3.98**	2.54 **3.86**	2.50 **3.78**	2.46 **3.70**	2.42 **3.61**	2.40 **3.56**	2.36 **3.49**	2.35 **3.46**	2.32 **3.41**	2.31 **3.38**	2.30 **3.36**	12
2.55 **3.85**	2.51 **3.78**	2.46 **3.67**	2.42 **3.59**	2.38 **3.51**	2.34 **3.42**	2.32 **3.37**	2.28 **3.30**	2.26 **3.27**	2.24 **3.21**	2.22 **3.18**	2.21 **3.16**	13
2.48 **3.70**	2.44 **3.62**	2.39 **3.51**	2.35 **3.43**	2.31 **3.34**	2.27 **3.26**	2.24 **3.21**	2.21 **3.14**	2.19 **3.11**	2.16 **3.06**	2.14 **3.02**	2.13 **3.00**	14
2.43 **3.56**	2.39 **3.48**	2.33 **3.36**	2.29 **3.29**	2.25 **3.20**	2.21 **3.12**	2.18 **3.07**	2.15 **3.00**	2.12 **2.97**	2.10 **2.92**	2.08 **2.89**	2.07 **2.87**	15
2.37 **3.45**	2.33 **3.37**	2.28 **3.25**	2.24 **3.18**	2.20 **3.10**	2.16 **3.01**	2.13 **2.96**	2.09 **2.89**	2.07 **2.86**	2.04 **2.80**	2.02 **2.77**	2.01 **2.75**	16
2.33 **3.35**	2.29 **3.27**	2.23 **3.16**	2.19 **3.08**	2.15 **3.00**	2.11 **2.92**	2.08 **2.86**	2.04 **2.79**	2.02 **2.76**	1.99 **2.70**	1.97 **2.67**	1.96 **2.65**	17
2.29 **3.27**	2.25 **3.19**	2.19 **3.07**	2.15 **3.00**	2.11 **2.91**	2.07 **2.83**	2.04 **2.78**	2.00 **2.71**	1.98 **2.68**	1.95 **2.62**	1.93 **2.59**	1.92 **2.57**	18

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f_1 Degrees of Freedom (for

f_2	1	2	3	4	5	6	7	8	9	10	11	12
19	4.38 **8.18**	3.52 **5.93**	3.13 **5.01**	2.90 **4.50**	2.74 **4.17**	2.63 **3.94**	2.55 **3.77**	2.48 **3.63**	2.43 **3.52**	2.38 **3.43**	2.34 **3.36**	2.31 **3.30**
20	4.35 **8.10**	3.49 **5.85**	3.10 **4.94**	2.87 **4.43**	2.71 **4.10**	2.60 **3.87**	2.52 **3.71**	2.45 **3.56**	2.40 **3.45**	2.35 **3.37**	2.31 **3.30**	2.28 **3.23**
21	4.32 **8.02**	3.47 **5.78**	3.07 **4.87**	2.84 **4.37**	2.68 **4.04**	2.57 **3.81**	2.49 **3.65**	2.42 **3.51**	2.37 **3.40**	2.32 **3.31**	2.28 **3.24**	2.25 **3.17**
22	4.30 **7.94**	3.44 **5.72**	3.05 **4.82**	2.82 **4.31**	2.66 **3.99**	2.55 **3.76**	2.47 **3.59**	2.40 **3.45**	2.35 **3.35**	2.30 **3.26**	2.26 **3.18**	2.23 **3.12**
23	4.28 **7.88**	3.42 **5.66**	3.03 **4.76**	2.80 **4.26**	2.64 **3.94**	2.53 **3.71**	2.45 **3.54**	2.38 **3.41**	2.32 **3.30**	2.28 **3.21**	2.24 **3.14**	2.20 **3.07**
24	4.26 **7.82**	3.40 **5.61**	3.01 **4.72**	2.78 **4.22**	2.62 **3.90**	2.51 **3.67**	2.43 **3.50**	2.36 **3.36**	2.30 **3.25**	2.26 **3.17**	2.22 **3.09**	2.18 **3.03**
25	4.24 **7.77**	3.38 **5.57**	2.99 **4.68**	2.76 **4.18**	2.60 **3.86**	2.49 **3.63**	2.41 **3.46**	2.34 **3.32**	2.28 **3.21**	2.24 **3.13**	2.20 **3.05**	2.16 **2.99**
26	4.22 **7.72**	3.37 **5.53**	2.98 **4.64**	2.74 **4.14**	2.59 **3.82**	2.47 **3.59**	2.39 **3.42**	2.32 **3.29**	2.27 **3.17**	2.22 **3.09**	2.18 **3.02**	2.15 **2.96**
27	4.21 **7.68**	3.35 **5.49**	2.96 **4.60**	2.73 **4.11**	2.57 **3.79**	2.46 **3.56**	2.37 **3.39**	2.30 **3.26**	2.25 **3.14**	2.20 **3.06**	2.16 **2.98**	2.13 **2.93**
28	4.20 **7.64**	3.34 **5.45**	2.95 **4.57**	2.71 **4.07**	2.56 **3.76**	2.44 **3.53**	2.36 **3.36**	2.29 **3.23**	2.24 **3.11**	2.19 **3.03**	2.15 **2.95**	2.12 **2.90**
29	4.18 **7.60**	3.33 **5.42**	2.93 **4.54**	2.70 **4.04**	2.54 **3.73**	2.43 **3.50**	2.35 **3.33**	2.28 **3.20**	2.22 **3.08**	2.18 **3.00**	2.14 **2.92**	2.10 **2.87**
30	4.17 **7.56**	3.32 **5.39**	2.92 **4.51**	2.69 **4.02**	2.53 **3.70**	2.42 **3.47**	2.34 **3.30**	2.27 **3.17**	2.21 **3.06**	2.16 **2.98**	2.12 **2.90**	2.09 **2.84**
32	4.15 **7.50**	3.30 **5.34**	2.90 **4.46**	2.67 **3.97**	2.51 **3.66**	2.40 **3.42**	2.32 **3.25**	2.25 **3.12**	2.19 **3.01**	2.14 **2.94**	2.10 **2.86**	2.07 **2.80**
34	4.13 **7.44**	3.28 **5.29**	2.88 **4.42**	2.65 **3.93**	2.49 **3.61**	2.38 **3.38**	2.30 **3.21**	2.23 **3.08**	2.17 **2.97**	2.12 **2.89**	2.08 **2.82**	2.05 **2.76**
36	4.11 **7.39**	3.26 **5.25**	2.86 **4.38**	2.63 **3.89**	2.48 **3.58**	2.36 **3.35**	2.28 **3.18**	2.21 **3.04**	2.15 **2.94**	2.10 **2.86**	2.06 **2.78**	2.03 **2.72**
38	4.10 **7.35**	3.25 **5.21**	2.85 **4.34**	2.62 **3.86**	2.46 **3.54**	2.35 **3.32**	2.26 **3.15**	2.19 **3.02**	2.14 **2.91**	2.09 **2.82**	2.05 **2.75**	2.02 **2.69**
40	4.08 **7.31**	3.23 **5.18**	2.84 **4.31**	2.61 **3.83**	2.45 **3.51**	2.34 **3.29**	2.25 **3.12**	2.18 **2.99**	2.12 **2.88**	2.07 **2.80**	2.04 **2.73**	2.00 **2.66**
42	4.07 **7.27**	3.22 **5.15**	2.83 **4.29**	2.59 **3.80**	2.44 **3.49**	2.32 **3.26**	2.24 **3.10**	2.17 **2.96**	2.11 **2.86**	2.06 **2.77**	2.02 **2.70**	1.99 **2.64**
44	4.06 **7.24**	3.21 **5.12**	2.82 **4.26**	2.58 **3.78**	2.43 **3.46**	2.31 **3.24**	2.23 **3.07**	2.16 **2.94**	2.10 **2.84**	2.05 **2.75**	2.01 **2.68**	1.98 **2.62**

(*Continued*)

Greater Mean Square)												
14	16	20	24	30	40	50	75	100	200	500	∞	f_2
2.26	2.21	2.15	2.11	2.07	2.02	2.00	1.96	1.94	1.91	1.90	1.88	19
3.19	**3.12**	**3.00**	**2.92**	**2.84**	**2.76**	**2.70**	**2.63**	**2.60**	**2.54**	**2.51**	**2.49**	
2.23	2.18	2.12	2.08	2.04	1.99	1.96	1.92	1.90	1.87	1.85	1.84	20
3.13	**3.05**	**2.94**	**2.86**	**2.77**	**2.69**	**2.63**	**2.56**	**2.53**	**2.47**	**2.44**	**2.42**	
2.20	2.15	2.09	2.05	2.00	1.96	1.93	1.89	1.87	1.84	1.82	1.81	21
3.07	**2.99**	**2.88**	**2.80**	**2.72**	**2.63**	**2.58**	**2.51**	**2.47**	**2.42**	**2.38**	**2.36**	
2.18	2.13	2.07	2.03	1.98	1.93	1.91	1.87	1.84	1.81	1.80	1.78	22
3.02	**2.94**	**2.83**	**2.75**	**2.67**	**2.58**	**2.53**	**2.46**	**2.42**	**2.37**	**2.33**	**2.31**	
2.14	2.10	2.04	2.00	1.96	1.91	1.88	1.84	1.82	1.79	1.77	1.76	23
2.97	**2.89**	**2.78**	**2.70**	**2.62**	**2.53**	**2.48**	**2.41**	**2.37**	**2.32**	**2.28**	**2.26**	
2.13	2.09	2.02	1.98	1.94	1.89	1.86	1.82	1.80	1.76	1.74	1.73	24
2.93	**2.85**	**2.74**	**2.66**	**2.58**	**2.49**	**2.44**	**2.36**	**2.33**	**2.27**	**2.23**	**2.21**	
2.11	2.06	2.00	1.96	1.92	1.87	1.84	1.80	1.77	1.74	1.72	1.71	25
2.89	**2.81**	**2.70**	**2.62**	**2.54**	**2.45**	**2.40**	**2.32**	**2.29**	**2.23**	**2.19**	**2.17**	
2.10	2.05	1.99	1.95	1.90	1.85	1.82	1.78	1.76	1.72	1.70	1.69	26
2.86	**2.77**	**2.66**	**2.58**	**2.50**	**2.41**	**2.36**	**2.28**	**2.25**	**2.19**	**2.15**	**2.13**	
2.08	2.03	1.97	1.93	1.88	1.84	1.80	1.76	1.74	1.71	1.68	1.67	27
2.83	**2.74**	**2.63**	**2.55**	**2.47**	**2.38**	**2.33**	**2.25**	**2.21**	**2.16**	**2.12**	**2.10**	
2.06	2.02	1.96	1.91	1.87	1.81	1.78	1.75	1.72	1.69	1.67	1.65	28
2.80	**2.71**	**2.60**	**2.52**	**2.44**	**2.35**	**2.30**	**2.22**	**2.18**	**2.13**	**2.09**	**2.06**	
2.05	2.00	1.94	1.90	1.85	1.80	1.77	1.73	1.71	1.68	1.65	1.64	29
2.77	**2.68**	**2.57**	**2.49**	**2.41**	**2.32**	**2.27**	**2.19**	**2.15**	**2.10**	**2.06**	**2.03**	
2.04	1.99	1.93	1.89	1.84	1.79	1.76	1.72	1.69	1.66	1.64	1.62	30
2.74	**2.66**	**2.55**	**2.47**	**2.38**	**2.29**	**2.24**	**2.16**	**2.13**	**2.07**	**2.03**	**2.01**	
2.02	1.97	1.91	1.86	1.82	1.76	1.74	1.69	1.67	1.64	1.61	1.59	32
2.70	**2.62**	**2.51**	**2.42**	**2.34**	**2.25**	**2.20**	**2.12**	**2.08**	**2.02**	**1.98**	**1.96**	
2.00	1.95	1.89	1.84	1.80	1.74	1.71	1.67	1.64	1.61	1.59	1.57	34
2.66	**2.58**	**2.47**	**2.38**	**2.30**	**2.21**	**2.15**	**2.08**	**2.04**	**1.98**	**1.94**	**1.91**	
1.98	1.93	1.87	1.82	1.78	1.72	1.69	1.65	1.62	1.59	1.56	1.55	36
2.62	**2.54**	**2.43**	**2.35**	**2.26**	**2.17**	**2.12**	**2.04**	**2.00**	**1.94**	**1.90**	**1.87**	
1.96	1.92	1.85	1.80	1.76	1.71	1.67	1.63	1.60	1.57	1.54	1.53	38
2.59	**2.51**	**2.40**	**2.32**	**2.22**	**2.14**	**2.08**	**2.00**	**1.97**	**1.90**	**1.86**	**1.84**	
1.95	1.90	1.84	1.79	1.74	1.69	1.66	1.61	1.59	1.55	1.53	1.51	40
2.56	**2.49**	**2.37**	**2.29**	**2.20**	**2.11**	**2.05**	**1.97**	**1.94**	**1.88**	**1.84**	**1.81**	
1.94	1.89	1.82	1.78	1.73	1.68	1.64	1.60	1.57	1.54	1.51	1.49	42
2.54	**2.46**	**2.35**	**2.26**	**2.17**	**2.08**	**2.02**	**1.94**	**1.91**	**1.85**	**1.80**	**1.78**	
1.92	1.88	1.81	1.76	1.72	1.66	1.63	1.58	1.56	1.52	1.50	1.48	44
2.52	**2.44**	**2.32**	**2.24**	**2.15**	**2.06**	**2.00**	**1.92**	**1.88**	**1.82**	**1.78**	**1.75**	

									f_1 Degrees of Freedom (for			
f_2	1	2	3	4	5	6	7	8	9	10	11	12
46	4.05 **7.21**	3.20 **5.10**	2.81 **4.24**	2.57 **3.76**	2.42 **3.44**	2.30 **3.22**	2.22 **3.05**	2.14 **2.92**	2.09 **2.82**	2.04 **2.73**	2.00 **2.66**	1.97 **2.60**
48	4.04 **7.19**	3.19 **5.08**	2.80 **4.22**	2.56 **3.74**	2.41 **3.42**	2.30 **3.20**	2.21 **3.04**	2.14 **2.90**	2.08 **2.80**	2.03 **2.71**	1.99 **2.64**	1.96 **2.58**
50	4.03 **7.17**	3.18 **5.06**	2.79 **4.20**	2.56 **3.72**	2.40 **3.41**	2.29 **3.18**	2.20 **3.02**	2.13 **2.88**	2.07 **2.78**	2.02 **2.70**	1.98 **2.62**	1.95 **2.56**
55	4.02 **7.12**	3.17 **5.01**	2.78 **4.16**	2.54 **3.68**	2.38 **3.37**	2.27 **3.15**	2.18 **2.98**	2.11 **2.85**	2.05 **2.75**	2.00 **2.66**	1.97 **2.59**	1.93 **2.53**
60	4.00 **7.08**	3.15 **4.98**	2.76 **4.13**	2.52 **3.65**	2.37 **3.34**	2.25 **3.12**	2.17 **2.95**	2.10 **2.82**	2.04 **2.72**	1.99 **2.63**	1.95 **2.56**	1.92 **2.50**
65	3.99 **7.04**	3.14 **4.95**	2.75 **4.10**	2.51 **3.62**	2.36 **3.31**	2.24 **3.09**	2.15 **2.93**	2.08 **2.79**	2.02 **2.70**	1.98 **2.61**	1.94 **2.54**	1.90 **2.47**
70	3.98 **7.01**	3.13 **4.92**	2.74 **4.08**	2.50 **3.60**	2.35 **3.29**	2.23 **3.07**	2.14 **2.91**	2.07 **2.77**	2.01 **2.67**	1.97 **2.59**	1.93 **2.51**	1.89 **2.45**
80	3.96 **6.96**	3.11 **4.88**	2.72 **4.04**	2.48 **3.56**	2.33 **3.25**	2.21 **3.04**	2.12 **2.87**	2.05 **2.74**	1.99 **2.64**	1.95 **2.55**	1.91 **2.48**	1.88 **2.41**
100	3.94 **6.90**	3.09 **4.82**	2.70 **3.98**	2.46 **3.51**	2.30 **3.20**	2.19 **2.99**	2.10 **2.82**	2.03 **2.69**	1.97 **2.59**	1.92 **2.51**	1.88 **2.43**	1.85 **2.36**
125	3.92 **6.84**	3.07 **4.78**	2.68 **3.94**	2.44 **3.47**	2.29 **3.17**	2.17 **2.95**	2.08 **2.79**	2.01 **2.65**	1.95 **2.56**	1.90 **2.47**	1.86 **2.40**	1.83 **2.33**
150	3.91 **6.81**	3.06 **4.75**	2.67 **3.91**	2.43 **3.44**	2.27 **3.14**	2.16 **2.92**	2.07 **2.76**	2.00 **2.62**	1.94 **2.53**	1.89 **2.44**	1.85 **2.37**	1.82 **2.30**
200	3.89 **6.76**	3.04 **4.71**	2.65 **3.88**	2.41 **3.41**	2.26 **3.11**	2.14 **2.90**	2.05 **2.73**	1.98 **2.60**	1.92 **2.50**	1.87 **2.41**	1.83 **2.34**	1.80 **2.28**
400	3.86 **6.70**	3.02 **4.66**	2.62 **3.83**	2.39 **3.36**	2.23 **3.06**	2.12 **2.85**	2.03 **2.69**	1.96 **2.55**	1.90 **2.46**	1.85 **2.37**	1.81 **2.29**	1.78 **2.23**
1000	3.85 **6.66**	3.00 **4.62**	2.61 **3.80**	2.38 **3.34**	2.22 **3.04**	2.10 **2.82**	2.02 **2.66**	1.95 **2.53**	1.89 **2.43**	1.84 **2.34**	1.80 **2.26**	1.76 **2.20**
∞	3.84 **6.64**	2.99 **4.60**	2.60 **3.78**	2.37 **3.32**	2.21 **3.02**	2.09 **2.80**	2.01 **2.64**	1.94 **2.51**	1.88 **2.41**	1.83 **2.32**	1.79 **2.24**	1.75 **2.18**

(*Continued*)

Greater Mean Square)												
14	16	20	24	30	40	50	75	100	200	500	∞	f_2
1.91 **2.50**	1.87 **2.42**	1.80 **2.30**	1.75 **2.22**	1.71 **2.13**	1.65 **2.04**	1.62 **1.98**	1.57 **1.90**	1.54 **1.86**	1.51 **1.80**	1.48 **1.76**	1.46 **1.72**	46
1.90 **2.48**	1.86 **2.40**	1.79 **2.28**	1.74 **2.20**	1.70 **2.11**	1.64 **2.02**	1.61 **1.96**	1.56 **1.88**	1.53 **1.84**	1.50 **1.78**	1.47 **1.73**	1.45 **1.70**	48
1.90 **2.46**	1.85 **2.39**	1.78 **2.26**	1.74 **2.18**	1.69 **2.10**	1.63 **2.00**	1.60 **1.94**	1.55 **1.86**	1.52 **1.82**	1.48 **1.76**	1.46 **1.71**	1.44 **1.68**	50
1.88 **2.43**	1.83 **2.35**	1.76 **2.23**	1.72 **2.15**	1.67 **2.06**	1.61 **1.96**	1.58 **1.90**	1.52 **1.82**	1.50 **1.78**	1.46 **1.71**	1.43 **1.66**	1.41 **1.64**	55
1.86 **2.40**	1.81 **2.32**	1.75 **2.20**	1.70 **2.12**	1.65 **2.03**	1.59 **1.93**	1.56 **1.87**	1.50 **1.79**	1.48 **1.74**	1.44 **1.68**	1.41 **1.63**	1.39 **1.60**	60
1.85 **2.37**	1.80 **2.30**	1.73 **2.18**	1.68 **2.09**	1.63 **2.00**	1.57 **1.90**	1.54 **1.84**	1.49 **1.76**	1.46 **1.71**	1.42 **1.64**	1.39 **1.60**	1.37 **1.56**	65
1.84 **2.35**	1.79 **2.28**	1.72 **2.15**	1.67 **2.07**	1.62 **1.98**	1.56 **1.88**	1.53 **1.82**	1.47 **1.74**	1.45 **1.69**	1.40 **1.62**	1.37 **1.56**	1.35 **1.53**	70
1.82 **2.32**	1.77 **2.24**	1.70 **2.11**	1.65 **2.03**	1.60 **1.94**	1.54 **1.84**	1.51 **1.78**	1.45 **1.70**	1.42 **1.65**	1.38 **1.57**	1.35 **1.52**	1.32 **1.49**	80
1.79 **2.26**	1.75 **2.19**	1.68 **2.06**	1.63 **1.98**	1.57 **1.89**	1.51 **1.79**	1.48 **1.73**	1.42 **1.64**	1.39 **1.59**	1.34 **1.51**	1.30 **1.46**	1.28 **1.43**	100
1.77 **2.23**	1.72 **2.15**	1.65 **2.03**	1.60 **1.94**	1.55 **1.85**	1.49 **1.75**	1.45 **1.68**	1.39 **1.59**	1.36 **1.54**	1.31 **1.46**	1.27 **1.40**	1.25 **1.37**	125
1.76 **2.20**	1.71 **2.12**	1.64 **2.00**	1.59 **1.91**	1.54 **1.83**	1.47 **1.72**	1.44 **1.66**	1.37 **1.56**	1.34 **1.51**	1.29 **1.43**	1.25 **1.37**	1.22 **1.33**	150
1.74 **2.17**	1.69 **2.09**	1.62 **1.97**	1.57 **1.88**	1.52 **1.79**	1.45 **1.69**	1.42 **1.62**	1.35 **1.53**	1.32 **1.48**	1.26 **1.39**	1.22 **1.33**	1.19 **1.28**	200
1.72 **2.12**	1.67 **2.04**	1.60 **1.92**	1.54 **1.84**	1.49 **1.74**	1.42 **1.64**	1.38 **1.57**	1.32 **1.47**	1.28 **1.42**	1.22 **1.32**	1.16 **1.24**	1.13 **1.19**	400
1.70 **2.09**	1.65 **2.01**	1.58 **1.89**	1.53 **1.81**	1.47 **1.71**	1.41 **1.61**	1.36 **1.54**	1.30 **1.44**	1.26 **1.38**	1.19 **1.28**	1.13 **1.19**	1.08 **1.11**	1000
1.69 **2.07**	1.64 **1.99**	1.57 **1.87**	1.52 **1.79**	1.46 **1.69**	1.40 **1.59**	1.35 **1.52**	1.28 **1.41**	1.24 **1.36**	1.17 **1.25**	1.11 **1.15**	1.00 **1.00**	∞

TABLE VI. Values of r for Different Levels of Significance

n	.1	.05	.02	.01	.001
1	.98769	.99692	.999507	.999877	.9999988
2	.90000	.95000	.98000	.990000	.99900
3	.8054	.8783	.93433	.95873	.99116
4	.7293	.8114	.8822	.91720	.97406
5	.6694	.7545	.8329	.8745	.95074
6	.6215	.7067	.7887	.8343	.92493
7	.5822	.6664	.7498	.7977	.8982
8	.5494	.6319	.7155	.7646	.8721
9	.5214	.6021	.6851	.7348	.8471
10	.4973	.5760	.6581	.7079	.8233
11	.4762	.5529	.6339	.6835	.8010
12	.4575	.5324	.6120	.6614	.7800
13	.4409	.5139	.5923	.6411	.7603
14	.4259	.4973	.5742	.6226	.7420
15	.4124	.4821	.5577	.6055	.7246
16	.4000	.4683	.5425	.5897	.7084
17	.3887	.4555	.5285	.5751	.6932
18	.3783	.4438	.5155	.5614	.6787
19	.3687	.4329	.5034	.5487	.6652
20	.3598	.4227	.4921	.5368	.6524
25	.3233	.3809	.4451	.4869	.5974
30	.2960	.3494	.4093	.4487	.5541
35	.2746	.3246	.3810	.4182	.5189
40	.2573	.3044	.3578	.3932	.4896
45	.2428	.2875	.3384	.3721	.4648
50	.2306	.2732	.3218	.3541	.4433
60	.2108	.2500	.2948	.3248	.4078
70	.1954	.2319	.2737	.3017	.3799
80	.1829	.2172	.2565	.2830	.3568
90	.1726	.2050	.2422	.2673	.3375
100	.1638	.1946	.2301	.2540	.3211

SOURCE: Table VI is reprinted from Table VI of R. A. Fisher and F. Yates: *Statistical Tables for Biological, Agricultural, and Medical Research*, published by Oliver and Boyd Ltd., Edinburgh, by permission of the authors and publishers.

TABLE VII. Table of z Values for r[a]

r	z	r	z	r	z	r	z	r	z
.000	.000	.200	.203	.400	.424	.600	.693	.800	1.099
.005	.005	.205	.208	.405	.430	.605	.701	.805	1.113
.010	.010	.210	.213	.410	.436	.610	.709	.810	1.127
.015	.015	.215	.218	.415	.442	.615	.717	.815	1.142
.020	.020	.220	.224	.420	.448	.620	.725	.820	1.157
.025	.025	.225	.229	.425	.454	.625	.733	.825	1.172
.030	.030	.230	.234	.430	.460	.630	.741	.830	1.188
.035	.035	.235	.239	.435	.466	.635	.750	.835	1.204
.040	.040	.240	.245	.440	.472	.640	.758	.840	1.221
.045	.045	.245	.250	.445	.478	.645	.767	.845	1.238
.050	.050	.250	.255	.450	.485	.650	.775	.850	1.256
.055	.055	.255	.261	.455	.491	.655	.784	.855	1.274
.060	.060	.260	.266	.460	.497	.660	.793	.860	1.293
.065	.065	.265	.271	.465	.504	.665	.802	.865	1.313
.070	.070	.270	.277	.470	.510	.670	.811	.870	1.333
.075	.075	.275	.282	.475	.517	.675	.820	.875	1.354
.080	.080	.280	.288	.480	.523	.680	.829	.880	1.376
.085	.085	.285	.293	.485	.530	.685	.838	.885	1.398
.090	.090	.290	.299	.490	.536	.690	.848	.890	1.422
.095	.095	.295	.304	.495	.543	.695	.858	.895	1.447
.100	.100	.300	.310	.500	.549	.700	.867	.900	1.472
.105	.105	.305	.315	.505	.556	.705	.877	.905	1.499
.110	.110	.310	.321	.510	.563	.710	.887	.910	1.528
.115	.116	.315	.326	.515	.570	.715	.897	.915	1.557
.120	.121	.320	.332	.520	.576	.720	.908	.920	1.589
.125	.126	.325	.337	.525	.583	.725	.918	.925	1.623
.130	.131	.330	.343	.530	.590	.730	.929	.930	1.658
.135	.136	.335	.348	.535	.597	.735	.940	.935	1.697
.140	.141	.340	.354	.540	.604	.740	.950	.940	1.738
.145	.146	.345	.360	.545	.611	.745	.962	.945	1.783
.150	.151	.350	.365	.550	.618	.750	.973	.950	1.832
.155	.156	.355	.371	.555	.626	.755	.984	.955	1.886
.160	.161	.360	.377	.560	.633	.760	.996	.960	1.946
.165	.167	.365	.383	.565	.640	.765	1.008	.965	2.014
.170	.172	.370	.388	.570	.648	.770	1.020	.970	2.092
.175	.177	.375	.394	.575	.655	.775	1.033	.975	2.185
.180	.182	.380	.400	.580	.662	.780	1.045	.980	2.298
.185	.187	.385	.406	.585	.670	.785	1.058	.985	2.443
.190	.192	.390	.412	.590	.678	.790	1.071	.990	2.647
.195	.198	.395	.418	.595	.685	.795	1.085	.995	2.994

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[a] Table VII was constructed by F. P. Kilpatrick and D. A. Buchanan from the formula

$$z = {}^1/_2[\log_e(1 + r) - \log_e(1 - r)]$$

SOURCE: A. L. Edwards. *Statistical Methods for the Behavioral Sciences.* New York: Rinehart, 1954. Reprinted by permission of the publisher.

TABLE VIII. Estimates of r_{tet} for Various Values of ad/bc

r_{tet}	ad/bc	r_{tet}	ad/bc	r_{tet}	ad/bc
.00	0–1.00	.35	2.49–2.55	.70	8.50–8.90
.01	1.01–1.03	.36	2.56–2.63	.71	8.91–9.35
.02	1.04–1.06	.37	2.64–2.71	.72	9.36–9.82
.03	1.07–1.08	.38	2.72–2.79	.73	9.83–10.33
.04	1.09–1.11	.39	2.80–2.87	.74	10.34–10.90
.05	1.12–1.14	.40	2.88–2.96	.75	10.91–11.51
.06	1.15–1.17	.41	2.97–3.05	.76	11.52–12.16
.07	1.18–1.20	.42	3.06–3.14	.77	12.17–12.89
.08	1.21–1.23	.43	3.15–3.24	.78	12.90–13.70
.09	1.24–1.27	.44	3.25–3.34	.79	13.71–14.58
.10	1.28–1.30	.45	3.35–3.45	.80	14.59–15.57
.11	1.31–1.33	.46	3.46–3.56	.81	15.58–16.65
.12	1.34–1.37	.47	3.57–3.68	.82	16.66–17.88
.13	1.38–1.40	.48	3.69–3.80	.83	17.89–19.28
.14	1.41–1.44	.49	3.81–3.92	.84	19.29–20.85
.15	1.45–1.48	.50	3.93–4.06	.85	20.86–22.68
.16	1.49–1.52	.51	4.07–4.20	.86	22.69–24.76
.17	1.53–1.56	.52	4.21–4.34	.87	24.77–27.22
.18	1.57–1.60	.53	4.35–4.49	.88	27.23–30.09
.19	1.61–1.64	.54	4.50–4.66	.89	30.10–33.60
.20	1.65–1.69	.55	4.67–4.82	.90	33.61–37.79
.21	1.70–1.73	.56	4.83–4.99	.91	37.80–43.06
.22	1.74–1.78	.57	5.00–5.18	.92	43.07–49.83
.23	1.79–1.83	.58	5.19–5.38	.93	49.84–58.79
.24	1.84–1.88	.59	5.39–5.59	.94	58.80–70.95
.25	1.89–1.93	.60	5.60–5.80	.95	70.96–89.01
.26	1.94–1.98	.61	5.81–6.03	.96	89.02–117.54
.27	1.99–2.04	.62	6.04–6.28	.97	117.55–169.67
.28	2.05–2.10	.63	6.29–6.54	.98	169.68–293.12
.29	2.11–2.15	.64	6.55–6.81	.99	293.13–923.97
.30	2.16–2.22	.65	6.82–7.10	1.00	923.98 . . .
.31	2.23–2.28	.66	7.11–7.42		
.32	2.29–2.34	.67	7.43–7.75		
.33	2.35–2.41	.68	7.76–8.11		
.34	2.42–2.48	.69	8.12–8.49		

SOURCE: M. D. Davidoff and H. W. Goheen. A table for the rapid determination of the tetrachoric correlation coefficient. *Psychometrika*, 1953, **18**, 115–121. Reprinted with the permission of the authors and publisher.

TABLE IX. Table of Critical Values of T in the Wilcoxon Matched-Pairs Signed-Ranks Test

	Level of Significance for One-Tailed Test		
N	.025	.01	.005
	Level of Significance for Two-Tailed Test		
	.05	.02	.01
6	0	—	—
7	2	0	—
8	4	2	0
9	6	3	2
10	8	5	3
11	11	7	5
12	14	10	7
13	17	13	10
14	21	16	13
15	25	20	16
16	30	24	20
17	35	28	23
18	40	33	28
19	46	38	32
20	52	43	38
21	59	49	43
22	66	56	49
23	73	62	55
24	81	69	61
25	89	77	68

SOURCE: Adapted from Table I of F. Wilcoxon. *Some Rapid Approximate Statistical Procedures.* New York: American Cyanamid Company, 1949, p. 13. Reproduced from S. Siegel. *Nonparametric Statistics for the Behavioral Sciences.* New York: McGraw-Hill, 1956. Reprinted by permission of the author, American Cyanamid Company, and McGraw-Hill Book Company.

TABLE X. Table of Critical Values of *U* in the Mann-Whitney Test

(*a*) Critical Values of *U* for a One-Tailed Test at .001 or for a Two-Tailed Test at .002

n_1 \ n_2	9	10	11	12	13	14	15	16	17	18	19	20
1												
2												
3									0	0	0	0
4		0	0	0	1	1	1	2	2	3	3	3
5	1	1	2	2	3	3	4	5	5	6	7	7
6	2	3	4	4	5	6	7	8	9	10	11	12
7	3	5	6	7	8	9	10	11	13	14	15	16
8	5	6	8	9	11	12	14	15	17	18	20	21
9	7	8	10	12	14	15	17	19	21	23	25	26
10	8	10	12	14	17	19	21	23	25	27	29	32
11	10	12	15	17	20	22	24	27	29	32	34	37
12	12	14	17	20	23	25	28	31	34	37	40	42
13	14	17	20	23	26	29	32	35	38	42	45	48
14	15	19	22	25	29	32	36	39	43	46	50	54
15	17	21	24	28	32	36	40	43	47	51	55	59
16	19	23	27	31	35	39	43	48	52	56	60	65
17	21	25	29	34	38	43	47	52	57	61	66	70
18	23	27	32	37	42	46	51	56	61	66	71	76
19	25	29	34	40	45	50	55	60	66	71	77	82
20	26	32	37	42	48	54	59	65	70	76	82	88

(*b*) Critical Values of *U* for a One-Tailed Test at .01 or for a Two-Tailed Test at .02

n_1 \ n_2	9	10	11	12	13	14	15	16	17	18	19	20
1												
2					0	0	0	0	0	0	1	1
3	1	1	1	2	2	2	3	3	4	4	4	5
4	3	3	4	5	5	6	7	7	8	9	9	10
5	5	6	7	8	9	10	11	12	13	14	15	16
6	7	8	9	11	12	13	15	16	18	19	20	22
7	9	11	12	14	16	17	19	21	23	24	26	28
8	11	13	15	17	20	22	24	26	28	30	32	34
9	14	16	18	21	23	26	28	31	33	36	38	40
10	16	19	22	24	27	30	33	36	38	41	44	47
11	18	22	25	28	31	34	37	41	44	47	50	53
12	21	24	28	31	35	38	42	46	49	53	56	60
13	23	27	31	35	39	43	47	51	55	59	63	67
14	26	30	34	38	43	47	51	56	60	65	69	73
15	28	33	37	42	47	51	56	61	66	70	75	80
16	31	36	41	46	51	56	61	66	71	76	82	87
17	33	38	44	49	55	60	66	71	77	82	88	93
18	36	41	47	53	59	65	70	76	82	88	94	100
19	38	44	50	56	63	69	75	82	88	94	101	107
20	40	47	53	60	67	73	80	87	93	100	107	114

SOURCE: Adapted and abridged from Tables 1, 3, 5, and 7 of D. Aube. Extended tables for the Mann-Whitney statistic. *Bulletin of the Institute of Educational Research at Indiana University*, 1953, **1**, No. 2. Reproduced from S. Siegel. *Nonparametric Statistics for the Behavioral Sciences*. New York: McGraw-Hill, 1956. Reprinted by permission of the author, Institute of Educational Research, and McGraw-Hill Book Company.

TABLE X (*Continued*)

(*c*) Critical Values of *U* for a One-Tailed Test at .025 or for a Two-Tailed Test at .05

n_1 \ n_2	9	10	11	12	13	14	15	16	17	18	19	20
1												
2	0	0	0	1	1	1	1	1	2	2	2	2
3	2	3	3	4	4	5	5	6	6	7	7	8
4	4	5	6	7	8	9	10	11	11	12	13	13
5	7	8	9	11	12	13	14	15	17	18	19	20
6	10	11	13	14	16	17	19	21	22	24	25	27
7	12	14	16	18	20	22	24	26	28	30	32	34
8	15	17	19	22	24	26	29	31	34	36	38	41
9	17	20	23	26	28	31	34	37	39	42	45	48
10	20	23	26	29	33	36	39	42	45	48	52	55
11	23	26	30	33	37	40	44	47	51	55	58	62
12	26	29	33	37	41	45	49	53	57	61	65	69
13	28	33	37	41	45	50	54	59	63	67	72	76
14	31	36	40	45	50	55	59	64	67	74	78	83
15	34	39	44	49	54	59	64	70	75	80	85	90
16	37	42	47	53	59	64	70	75	81	86	92	98
17	39	45	51	57	63	67	75	81	87	93	99	105
18	42	48	55	61	67	74	80	86	93	99	106	112
19	45	52	58	65	72	78	85	92	99	106	113	119
20	48	55	62	69	76	83	90	98	105	112	119	127

(*d*) Critical Values of *U* for a One-Tailed Test at .05 or for a Two-Tailed Test at .10

n_2 \ n_1	9	10	11	12	13	14	15	16	17	18	19	20
1											0	0
2	1	1	1	2	2	2	3	3	3	4	4	4
3	3	4	5	5	6	7	7	8	9	9	10	11
4	6	7	8	9	10	11	12	14	15	16	17	18
5	9	11	12	13	15	16	18	19	20	22	23	25
6	12	14	16	17	19	21	23	25	26	28	30	32
7	15	17	19	21	24	26	28	30	33	35	37	39
8	18	20	23	26	28	31	33	36	39	41	44	47
9	21	24	27	30	33	36	39	42	45	48	51	54
10	24	27	31	34	37	41	44	48	51	55	58	62
11	27	31	34	38	42	46	50	54	57	61	65	69
12	30	34	38	42	47	51	55	60	64	68	72	77
13	33	37	42	47	51	56	61	65	70	75	80	84
14	36	41	46	51	56	61	66	71	77	82	87	92
15	39	44	50	55	61	66	72	77	83	88	94	100
16	42	48	54	60	65	71	77	83	89	95	101	107
17	45	51	57	64	70	77	83	89	96	102	109	115
18	48	55	61	68	75	82	88	95	102	109	116	123
19	51	58	65	72	80	87	94	101	109	116	123	130
20	54	62	69	77	84	92	100	107	115	123	130	138

TABLE XI. Table of Critical Values of r in the Runs Test

In the bodies of Tables XI(a) and XI(b) are various critical values of r for various values of n_1 and n_2. For the one-sample runs test, any value of r which is equal to or smaller than that shown in Table XI(a) or equal to or larger than that shown in Table XI(b) is significant at the .05 level. For the Wald-Wolfowitz two-sample runs test, any value of r which is equal to or smaller than that shown in Table XI(a) is significant at the .05 level.

(a)

n_2 n_1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2											2	2	2	2	2	2	2	2	2
3					2	2	2	2	2	2	2	2	2	3	3	3	3	3	3
4				2	2	2	3	3	3	3	3	3	3	3	4	4	4	4	4
5			2	2	3	3	3	3	3	4	4	4	4	4	4	4	5	5	5
6		2	2	3	3	3	3	4	4	4	4	5	5	5	5	5	5	6	6
7		2	2	3	3	3	4	4	5	5	5	5	5	6	6	6	6	6	6
8		2	3	3	3	4	4	5	5	5	6	6	6	6	6	7	7	7	7
9		2	3	3	4	4	5	5	5	6	6	6	7	7	7	7	8	8	8
10		2	3	3	4	5	5	5	6	6	7	7	7	7	8	8	8	8	9
11		2	3	4	4	5	5	6	6	7	7	7	8	8	8	9	9	9	9
12	2	2	3	4	4	5	6	6	7	7	7	8	8	8	9	9	9	10	10
13	2	2	3	4	5	5	6	6	7	7	8	8	9	9	9	10	10	10	10
14	2	2	3	4	5	5	6	7	7	8	8	9	9	9	10	10	10	11	11
15	2	3	3	4	5	6	6	7	7	8	8	9	9	10	10	11	11	11	12
16	2	3	4	4	5	6	6	7	8	8	9	9	10	10	11	11	11	12	12
17	2	3	4	4	5	6	7	7	8	9	9	10	10	11	11	11	12	12	13
18	2	3	4	5	5	6	7	8	8	9	9	10	10	11	11	12	12	13	13
19	2	3	4	5	6	6	7	8	8	9	10	10	11	11	12	12	13	13	13
20	2	3	4	5	6	6	7	8	9	9	10	10	11	12	12	13	13	13	14

(b)

n_2 n_1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
2																			
3																			
4				9	9														
5			9	10	10	11	11												
6			9	10	11	12	12	13	13	13	13								
7				11	12	13	13	14	14	14	14	15	15	15					
8				11	12	13	14	14	15	15	16	16	16	16	17	17	17	17	17
9					13	14	14	15	16	16	16	17	17	18	18	18	18	18	18
10					13	14	15	16	16	17	17	18	18	18	19	19	19	20	20
11					13	14	15	16	17	17	18	19	19	19	20	20	20	21	21
12					13	14	16	16	17	18	19	19	20	20	21	21	21	22	22
13						15	16	17	18	19	19	20	20	21	21	22	22	23	23
14						15	16	17	18	19	20	20	21	22	22	23	23	23	24
15						15	16	18	18	19	20	21	22	22	23	23	24	24	25
16							17	18	19	20	21	21	22	23	23	24	25	25	25
17							17	18	19	20	21	22	23	23	24	25	25	26	26
18							17	18	19	20	21	22	23	24	25	25	26	26	27
19							17	18	20	21	22	23	23	24	25	26	26	27	27
20							17	18	20	21	22	23	24	25	25	26	27	27	28

SOURCE: Adapted from Frieda S. Swed and C. Eisenhart. Tables for testing randomness of grouping in a sequence of alternatives. *Annals of Mathematical Statistics*, 1943, **14**, 83–86. Reproduced from S. Siegel. *Nonparametric Statistics for the Behavioral Sciences*. New York: McGraw-Hill, 1956. Reprinted by permission of the authors, Institute of Mathematical Statistics, and McGraw-Hill Book Company.

TABLE XII. Values of H for Three Samples Significant at the 10, 5, and 1 Percent Levels

Sample Sizes			Level		
N_1	N_2	N_3	.10	.05	.01
2	2	2	4.57		
3	2	1	4.29		
3	2	2	4.50	4.71	
3	3	1	4.57	5.14	
3	3	2	4.56	5.36	6.25
3	3	3	4.62	5.60	6.49
4	2	1	4.50		
4	2	2	4.46	5.33	
4	3	1	4.06	5.21	
4	3	2	4.51	5.44	6.30
4	3	3	4.70	5.73	6.75
4	4	1	4.17	4.79	6.67
4	4	2	4.55	5.45	6.87
4	4	3	4.55	5.60	7.14
4	4	4	4.65	5.69	7.54
5	2	1	4.20	5.00	
5	2	2	4.37	5.16	6.53
5	3	1	4.02	4.96	
5	3	2	4.49	5.25	6.82
5	3	3	4.53	5.44	6.98
5	4	1	3.99	4.99	6.84
5	4	2	4.52	5.27	7.12
5	4	3	4.55	5.63	7.40
5	4	4	4.62	5.62	7.74
5	5	1	4.11	5.13	6.84
5	5	2	4.51	5.25	7.27
5	5	3	4.55	5.63	7.54
5	5	4	4.52	5.64	7.79
5	5	5	4.56	5.66	7.98

SOURCE: Abridged from Table 6.1 of W. H. Kruskal and W. A. Wallis. Use of ranks on one-criterion variance analysis. *Journal of the American Statistical Association*, 1952, **47**, 584–621. Reproduced from M. W. Tate and R. C. Clelland. *Nonparametric and Shortcut Statistics*. Danville, Ill.: The Interstate Printers and Publishers, 1957. Reprinted by permission of the authors, Journal of the American Statistical Association, and The Interstate Printers and Publishers.

TABLE XIII. Values of the Coefficient of Concordance *W* Significant at the 20, 10, 5, and 1 Percent Levels

m	*a*	3	4	5	6	7*n*	8	9	10
3	.20	.78	.60	.53	.49	.47	.46	.45	.44
	.10		.73	.62	.58	.55	.53	.52	.51
	.05	1.00	.82	.71	.65	.62	.60	.58	.56
	.01		.96	.84	.77	.73	.70	.67	.65
4	.20	.56	.40	.38	.37	.36	.35	.34	.33
	.10	.75	.52	.47	.44	.42	.41	.40	.39
	.05	.81	.65	.54	.51	.48	.46	.45	.44
	.01	1.00	.80	.67	.62	.59	.56	.54	.52
5	.20	.36	.34	.30	.29	.28	.28	.27	.27
	.10	.52	.42	.38	.36	.34	.33	.32	.31
	.05	.64	.52	.44	.41	.39	.38	.36	.35
	.01	.84	.66	.56	.52	.49	.46	.44	.43
6	.20	.33	.27	.25	.24	.24	.23	.23	.23
	.10	.44	.36	.32	.30	.29	.28	.27	.26
	.05	.58	.42	.37	.35	.33	.32	.31	.30
	.01	.75	.56	.49	.45	.42	.40	.38	.37
7	.20	.27	.23	.22	.21	.20	.20	.20	.19
	.10	.39	.30	.27	.26	.25	.24	.23	.23
	.05	.51	.36	.32	.30	.29	.27	.26	.26
	.01	.63	.48	.43	.39	.36	.34	.33	.32
8	.20	.25	.20	.19	.18	.18	.17	.17	.17
	.10	.33	.26	.24	.23	.22	.21	.20	.20
	.05	.39	.32	.29	.27	.25	.24	.23	.23
	.01	.56	.43	.38	.35	.32	.31	.29	.28
9	.20	.20	.18	.17	.16	.16	.16	.15	.15
	.10	.31	.23	.21	.20	.19	.19	.18	.18
	.05	.35	.28	.26	.24	.23	.22	.21	.20
	.01	.48	.38	.34	.31	.29	.27	.26	.25
10	.20	.19	.16	.15	.15	.14	.14	.14	.13
	.10	.25	.21	.19	.18	.17	.17	.16	.16
	.05	.31	.25	.23	.21	.20	.20	.19	.18
	.01	.48	.35	.31	.28	.26	.25	.24	.23
12	.20	.14	.13	.13	.12	.12	.12	.11	.11
	.10	.19	.17	.16	.15	.15	.14	.14	.13
	.05	.25	.21	.19	.18	.17	.16	.16	.15
	.01	.36	.30	.26	.24	.22	.21	.20	.19
14	.20	.12	.11	.11	.10	.10	.10	.10	.10
	.10	.17	.15	.14	.13	.13	.12	.12	.12
	.05	.21	.18	.17	.16	.15	.14	.14	.13
	.01	.31	.26	.23	.21	.19	.18	.17	.17

SOURCE: Values at the left of the broken line were derived from Appendix Tables 5A, 5B, 5C, and 5D of M. G. Kendall. *Rank Correlation Methods.* London: Charles Griffin and Co. Ltd., 1948. Other values were obtained by the method described by Kendall, *ibid.*, p. 84. Reproduced from M. W. Tate and R. C. Clelland. *Nonparametric and Shortcut Statistics.* Danville, Ill.: The Interstate Printers and Publishers, 1957. Reprinted by permission of the author, Charles Griffin and Co., Ltd., and The Interstate Printers and Publishers.

TABLE XIII (*Continued*)

m	a	3	4	5	6	7^n	8	9	10
16	.20	.10	.10	.09	.09	.09	.09	.09	.08
	.10	.15	.13	.12	.12	.11	.11	.10	.10
	.05	.19	.16	.15	.14	.13	.12	.12	.12
	.01	.28	.23	.20	.18	.17	.16	.15	.15
18	.20	.09	.09	.08	.08	.08	.08	.08	.07
	.10	.13	.12	.11	.10	.10	.09	.09	.09
	.05	.17	.14	.13	.12	.11	.11	.11	.10
	.01	.25	.20	.18	.16	.15	.14	.14	.13
20	.20	.08	.08	.07	.07	.07	.07	.07	.07
	.10	.11	.10	.10	.09	.09	.08	.08	.08
	.05	.15	.13	.12	.11	.10	.10	.10	.09
	.01	.22	.18	.16	.15	.14	.13	.12	.11
25	.20	.07	.06	.06	.06	.06	.06	.05	.05
	.10	.09	.08	.08	.07	.07	.07	.07	.06
	.05	.12	.10	.09	.09	.08	.08	.08	.07
	.01	.18	.15	.13	.12	.11	.10	.10	.09
30	.20	.05	.05	.05	.05	.05	.05	.05	.04
	.10	.08	.07	.06	.06	.06	.06	.06	.05
	.05	.10	.09	.08	.07	.07	.07	.07	.06
	.01	.15	.12	.11	.10	.09	.09	.08	.08

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